Elementary Number Theory (TN410)

Exercises: Sheet #4

May 20, 2015

1. Use the Chebichev Theorem to prove that if $\psi(X) = \sum_{m\le X} \Lambda(n)$, then
$$X \ll \psi(X) \ll X \quad X \to \infty$$
2. Show that
$$\sum_{n\le X} \Lambda(n)\log n = \psi(X)\log X + O(X).$$
3. Prove the identity: $\Lambda(n) = -\sum_{d|n} \mu(d)\log d$
4. Prove the identity: $(\Lambda * \Lambda)(n) = \Lambda(n)\log n + \sum_{d|n} \mu(d)\log^2 d$.
5. Dimostrare l'identità
$$\mathrm{mcm}[1,2,3,\ldots,n] = e^{\psi(n)}.$$

Let $n, q \in \mathbb{N}$. The **Ramanujan's sum** is defined by the formula
$$c_q(n) := \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\tfrac{an}{q}\right)$$
where $e(z) := e^{2\pi i z}$. Prove the following properties:

6. Verificare esplicitamente che
$$c_1(n) = 1, \quad c_2(n) = \cos n\pi \qquad c_3(n) = 2\cos\tfrac{2}{3}n\pi, \quad c_4(n) = 2\cos\tfrac{1}{2}n\pi$$
$$c_5(n) = 2\cos\tfrac{2}{5}n\pi + 2\cos\tfrac{4}{5}n\pi \qquad c_6(n) = 2\cos\tfrac{1}{3}n\pi$$
$$c_7(n) = 2\cos\tfrac{2}{7}n\pi + 2\cos\tfrac{4}{7}n\pi + 2\cos\tfrac{6}{7}n\pi \qquad c_8(n) = 2\cos\tfrac{1}{4}n\pi + 2\cos\tfrac{3}{4}n\pi$$
$$c_9(n) = 2\cos\tfrac{2}{9}n\pi + 2\cos\tfrac{4}{9}n\pi + 2\cos\tfrac{8}{9}n\pi \qquad c_{10}(n) = 2\cos\tfrac{1}{5}n\pi + 2\cos\tfrac{3}{5}n\pi$$

Let $\eta_q(n) = \sum_{k=1}^{q} e(\frac{kn}{q})$. Prove that

7. $\eta_q(n) = \begin{cases} 0 & \text{if } q \nmid n \\ q & \text{if } q \mid n \end{cases} \qquad \eta_q(n) = \sum_{d\,|\,q} c_d(n), \qquad c_q(n) = \sum_{d\,|\,q} \mu\left(\frac{q}{d}\right)\eta_d(n)$
8. $c_q(n)$ is multiplicative in the following sense: if $(q, r) = 1$ then $c_q(n)c_r(n) = c_{qr}(n)$.
9. if p is a prime number,
$$c_p(n) = \begin{cases} -1 & \text{if } p \nmid n \\ \varphi(p) & \text{if } p \mid n \end{cases} \qquad c_{p^k}(n) = \begin{cases} 0 & \text{if } p^{k-1} \nmid n \\ -p^{k-1} & \text{if } p^{k-1} \mid n \text{ and } p^k \nmid n \\ \varphi(p^k) & \text{if } p^k \mid n \end{cases}$$
10. $c_q(n) = \sum_{d\,|\,(q,n)} \mu\left(\frac{q}{d}\right) d$ (formula di Kluyver - 1906)
11. $c_q(n) = \mu\left(\frac{q}{(q,n)}\right) \frac{\varphi(q)}{\varphi\left(\frac{q}{(q,n)}\right)}$ (formula di von Sterneck)