Soluzioni 11-AM4

Laura Di Gregorio

6 dicembre 2004

1. $\hat{f}(\xi) = \dfrac{1}{\sqrt{2\pi}} \dfrac{e^{-i\xi a} - e^{-i\xi b}}{i\xi}$;

2. $\hat{f}(\xi) = \dfrac{\sqrt{2}}{\sqrt{\pi}} \left(\dfrac{\sin\xi}{\xi} - \displaystyle\int_0^1 x \cos x\xi \, dx \right)$.

   Svolgendo l'integrale si ottiene

$$\begin{aligned} \hat{f}(\xi) &= \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\sin\xi}{\xi} - \frac{\sin\xi}{\xi} - \frac{\cos\xi}{\xi^2} + \frac{1}{\xi^2} \right) \\ &= \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1 - \cos\xi}{\xi^2}; \end{aligned}$$

3. Si calcoli la trasformata di Fourier di $e^{-|x|}$. Risulta

$$\widehat{e^{-|x|}} = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{1}{1+\xi^2}$$

   e dalla formula

$$\widehat{f(ax)}(\xi) = \frac{1}{a} \hat{f}(\xi/a)$$

   si ottiene facilmente che

$$\widehat{e^{-a|x|}}(\xi) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{a}{a^2 + \xi^2}.$$

4. Usando che $e^{-ix\xi} = \cos x\xi - i \sin x\xi$ si ha che

$$
\begin{aligned}
\hat{f}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\, e^{-ix\xi}\, dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{1} (1-x^2)\, \cos x\xi\, dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{1} \cos x\xi\, dx - \frac{\sqrt{2}}{\sqrt{\pi}} \int_{0}^{1} x^2 \cos x\xi\, dx \\
&= \frac{\sqrt{2}}{\sqrt{\pi}} \left(\frac{\sin \xi}{\xi} - \frac{\sin \xi}{\xi} - 2\frac{\cos \xi}{\xi^2} + 2 \int_{0}^{1} \cos x\xi\, dx \right) \\
&= \frac{2\sqrt{2}}{\xi^2 \sqrt{\pi}} \left(\frac{\sin \xi}{\xi} - \cos \xi \right) .
\end{aligned}
$$

5. Risulta

$$
\begin{aligned}
\sqrt{2\pi}\hat{f}(\xi) &= \int_{0}^{2\pi} \sin x\, e^{-ix\xi}\, dx \\
&= \left[\frac{\sin x}{-i\xi} e^{-ix\xi} \right]_{0}^{2\pi} + \int_{0}^{2\pi} \frac{e^{-ix\xi}}{i\xi} \cos x\, dx \\
&= \left[\frac{e^{-ix\xi}}{\xi^2} \cos x \right]_{0}^{2\pi} + \frac{1}{\xi^2} \int_{0}^{2\pi} e^{-ix\xi} \sin x\, dx \\
&= \frac{1}{\xi^2} (e^{-2\pi i\xi} - 1) + \frac{1}{\xi^2} \int_{0}^{2\pi} e^{-ix\xi} \sin x\, dx
\end{aligned}
$$

da cui segue che

$$
\int_{0}^{2\pi} \sin x\, e^{-ix\xi}\, dx = \frac{e^{-2\pi i\xi} - 1}{\xi^2 - 1}
$$

e dunque

$$
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{e^{-2\pi i\xi} - 1}{\xi^2 - 1} .
$$