Soluzioni 9-AM3

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1. Si consideri il cambio di coordinate

$$\begin{cases} x = a\rho \sin\varphi \cos\theta \\ y = b\rho \sin\varphi \sin\theta \\ z = c\rho \cos\varphi \end{cases}$$

con $(\varphi, \theta) \in (0, \pi) \times (0, 2\pi)$ e $0 < \rho < 1$.
Il determinante Jacobiano è

$$\left| \frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)} \right| = abc\rho^2 \sin\varphi$$

Dunque

$$\begin{aligned} \iiint_{\mathcal{D}} x \, dx \, dy \, dz &= a^3bc \int_0^{\pi} \sin^3\varphi d\varphi \int_0^{2\pi} \cos^2\theta d\theta \int_0^1 \rho^4 d\rho \\ &= \frac{a^3bc}{5} \int_0^{\pi} \sin^3\varphi d\varphi \int_0^{2\pi} \cos^2\theta d\theta \\ &= \frac{\pi a^3bc}{5} \int_0^{\pi} \sin^3\varphi d\varphi \\ &= \frac{\pi a^3bc}{5} \int_0^{\pi} (1 - \cos^2\varphi) d(-\cos\varphi) \\ &= \frac{\pi a^3bc}{5} \left(2 - \frac{2}{3}\right) = \frac{4}{15} a^3bc\pi \end{aligned}$$

2. La funzione integranda continua su $\overline{S}$ e S misurabile in quanto ∂S ha misura nulla quindi integrabile su S.

$$\iiint_S \frac{1}{(y+1)^3} dxdydz = \int_0^1 \int_0^1 \int_0^{x+z} \frac{1}{(y+1)^3} dydxdz$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^1\int_0^1 \left(\frac{1}{(x+z+1)^2}-1\right) dxdz \\
&= -\frac{1}{2}\int_0^1 \left(\frac{1}{z+2}+\frac{1}{z+1}-1\right) dz = \frac{1}{2}\log\frac{3}{4}+\frac{1}{2}
\end{aligned}
$$

3. Si ha che

$$z+\sqrt{x}\le y \Longleftrightarrow y-z\ge 0 \quad \text{e} \quad x\le (y-z)^2$$

dunque

$$D=\{(x,y,z)\in\mathbb{R}^3 \,:\, 0\le y\le 1,\ 0\le z\le y,\ 0\le x\le (y-z)^2\}.$$

Applicando Fubini:

$$
\begin{aligned}
\iiint_D \dots &= \int_0^1 f(y)\left[\int_0^y \left(\int_0^{(y-z)^2} dx\right) dz\right] dy \\
&= \int_0^1 f(y)\left[\int_0^y (y-z)^2 dz\right] dy \\
&= \int_0^1 f(y)\left[y^2 z+\frac{z^3}{3}-yz^2\right]_0^y dy = \int_0^1 f(y)\frac{y^3}{3}\, dy.
\end{aligned}
$$