Abstract

We consider a popular model of Random Games for n players.

Each player is given a finite number of strategies and the payoffs associated to different strategy profiles are assumed to be independent and identically distributed random variables.

We review some of the properties of such games with emphasis on the notion of Pure Nash Equilibrium (PNE).

We give a detailed proof of the familiar fact that the number of PNE converges to a Poisson random variable.

As in previous works on this matter, the latter convergence is obtained by means of the Chen–Stein method for Poisson convergence which is reviewed here in some detail.

Summary

Game theory is a branch of applied mathematics which is used in the social sciences like biology, computer science, philosophy and economy.

Game theory attempts to mathematically capture behavior in strategic situations, in which an individual's success in making choices depends on the choices of others.

While initially developed to analyze competitions in which one individual does better at another's expense, it has been expanded to treat a wide class of interactions, which are classified according to several criteria [6].

Traditional applications of game theory attempt to find equilibria in these games sets of strategies in which individuals are unlikely to change their behavior [9].

Many equilibrium concepts have been developed, most famously is the Nash equilibrium, in fact the idea of Nash equilibrium is one of the most powerful concepts in game theory.

Nash is an American mathematician who worked in game theory, differential geometry, serving as a Senior Research Mathematician at Princeton University [5]. He shared the 1994 Nobel Memorial Prize in Economic Sciences with game theorists Reinhard Selten and John Harsanyi.

A game consists of a number of players, say n, and each player can play one strategy out of a set of finite strategy.

Loosely speaking a Nash Equilibria (N.E.) is a set of strategy (strategy profile) if each represents a best response to the other strategies. So, if all the players are playing the strategies in a Nash equilibrium, they have no unilateral incentive to deviate, since their strategy is the best they can do given what others are doing [3].

Nash equilibria are usually divided into two types: Pure strategy Nash equilibria (P.N.E.) and Mixed strategy Nash equilibria (M.N.E.).

MNE refer to situations where a single player chooses its strategy according to some prescribed probability law.

We shall focus on the conceptually simpler notion of PNE. To describe this notion in detail let $\sigma = (\sigma_1, ..., \sigma_n)$ denote the strategy profile where each σ_i belongs to a finite set Ω_1 and let $\{u_i(\sigma)\}_{\sigma\in\Omega,\, i=1,...,n.}$ denote the payoff tables in our game ($u_i(\sigma)$ represents the pasyoff of the i-th player in the strategy profile σ).

We say that $\sigma \in (\Omega_1)^n = \Omega_n$ is a PNE if

$$u_i(\sigma) \geq \max_{\overline{\sigma}^{(i)}} u_i(\overline{\sigma}^{(i)})$$

where the max is over all $\sigma' \in \Omega_n$ such that $\sigma'_j = \sigma_j$ for every $j \neq i$.

John Nash won a Nobel prize proving that there is mixed equilibrium for every finite game. While Nash proved that every finite game has a Nash mixed equilibrium, but not all have pure strategy Nash equilibria.

For an example of a game that does not have a Nash equilibrium in pure strategies see Matching pennies in Chapter(3).

However, many games have pure strategy Nash equilibria, we can see the example of Prisoner's Dilemma in Chapter(3).
Further, games can have both pure strategy and mixed strategy equilibria.
We have discussed the setting of a finite games. A *Random Game* is a game where the payoff functions are *random variables*.
Here we consider in detail the *Indipendent Model*, i.e. the case where the payoff of the different players are indipendent random variables [10].
The indipendent model is defined as follows. Let $\Sigma = \{1, ..., n\}$ be a set of players and $\sigma = (\sigma_1, ..., \sigma_n) \in \Omega_n$ is the strategy profile, where $\sigma_i \in \Omega_1$ is the strategy of the i-th player. For simplicity (and without serius loss of generality) we shall only consider the case $\Omega_1 = \{\pm 1\}$, therefore $|\Omega_n| = 2^n$.
The payoff table is specified by functions $u_i : \Omega_n \to \Re$, $i \in N$.
In our model the $\{u_i(\sigma)\}_{\sigma \in \Omega,\ i=1,...,n.}$ will be chosen as i.i.d. random variables with common low μ on $\Re$. We shall assume that μ has no atom.

Definition 0.1. *Let $N = \{1, ..., n\}$ be the set of players, and $u_i(\sigma_1, ..., \sigma_n) \in \Re$ the vector of the payoff for the i-th player.*
A profile $\sigma = (\sigma_1, ..., \sigma_n)$ is a P.N.E. when for every player $i \in N$:

$$u_i(\sigma) \geq u_i(\sigma^{(i)}),$$

where $\sigma^{(i)}$ denotes the profile σ "flipped" at i, more precisely

$$(\sigma^{(i)})_j = \begin{cases} \sigma_j & if\, j \neq i \\ -\sigma_j & if\, j = i \end{cases}$$

As an example consider a game with three players.

Here $\sigma_i \in \{-1, +1\}$, so we have $|\Omega| = 2^3$ strategy profiles.

Therefore there are, for every strategy profile $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, three different variables: $u_i(\sigma_1, \sigma_2, \sigma_3)$ with $i = 1, 2, 3$.

Now we write all the three different payoff tables:

First player.	Second player.	Third player.
$u_1(-1; -1; -1) = X_1$	$u_2(-1; -1; -1) = Y_1$	$u_3(-1; -1; -1) = Z_1$
$u_1(-1; +1; -1) = X_2$	$u_2(-1; +1; -1) = Y_2$	$u_3(-1; +1; -1) = Z_2$
$u_1(-1; -1; +1) = X_3$	$u_2(-1; -1; +1) = Y_3$	$u_3(-1; -1; +1) = Z_3$
$u_1(+1; -1; -1) = X_4$	$u_2(+1; -1; -1) = Y_4$	$u_3(+1; -1; -1) = Z_4$
$u_1(-1; +1; +1) = X_5$	$u_2(-1; +1; +1) = Y_5$	$u_3(-1; +1; +1) = Z_5$
$u_1(+1; -1; +1) = X_6$	$u_2(+1; -1; +1) = Y_6$	$u_3(+1; -1; +1) = Z_6$
$u_1(+1; +1; -1) = X_7$	$u_2(+1; +1; -1) = Y_7$	$u_3(+1; +1; -1) = Z_7$
$u_1(+1; +1; +1) = X_8$	$u_2(+1; +1; +1) = Y_8$	$u_3(+1; +1; +1) = Z_8$

Analize, for example, the case with:

- First player chooses: $+1$
- Second player chooses: -1
- Third player chooses: $+1$

Therefore the vector is $\sigma = (+1, -1, +1)$ and the pay-offs are $u_1(\sigma) = X_6$, $u_2(\sigma) = Y_6$ and $u_3(\sigma) = Z_6$.

Now the question is: what is the probability that $\sigma = (+1, -1, +1)$ is a P.N.E.?

$\sigma = (+1, -1, +1)$ is a P.N.E. $\Leftrightarrow (X_6 \geq X_3, Y_6 \geq Y_8, Z_6 \geq Z_4)$, because every player can change only his strategy without touch the other players's game.

Passing to the probability and using the indipendence condition we can find:

$$\begin{aligned} P\left(\sigma\, is\, a\, P.N.E.\right) &= P\left(\{X_6 \geq X_3\} \cap \{Y_6 \geq Y_8\} \cap \{Z_6 \geq Z_4\}\right) \\ &= P\left(X_6 \geq X_3\right) P\left(Y_6 \geq Y_8\right) P\left(Z_6 \geq Z_4\right) \\ &= \frac{1}{2}\frac{1}{2}\frac{1}{2} = \frac{1}{2^3} \end{aligned}$$

If the law μ of the payoff has density on $\Re$ gives by φ, we have

$$\begin{aligned} P\left(X_i \geq Y_i\right) &= \int_{\Re}\int_{\Re} \varphi(x)\varphi(y)1_{\{x \geq y\}}dxdy \\ &= \int_{\Re}\int_{\Re} \varphi(x)\varphi(y)1_{\{x < y\}}dxdy \\ &= \frac{1}{2} \end{aligned}$$

because the integrals are symmetrical and from the beginning we suppose μ without atoms.

Proposition 0.1. *In the indipendent model we have, for every n, and for every $\sigma \in \Omega_n$*

$$P(\sigma\, is\, a\, P.N.E.) = \frac{1}{2^n}.$$

In particular if $W = \sum_{\sigma \in \Omega_n} 1_{\{\sigma\ is\ a\ P.N.E.\}}$ then

$$E\left[W\right] = 1.$$

Proof. As in the case with $n = 3$ above we have

$$\begin{aligned} P(\sigma \; is \; a \; P.N.E.) &= \prod_{i=1}^{n} P\left(X_i \geq Y_i\right) \\ &= \prod_{i=1}^{n} \left\{ \int_{\Re} \int_{\Re} \varphi(x)\varphi(y) 1_{\{x \geq y\}} dxdy \right\} \\ &= \prod_{i=1}^{n} \frac{1}{2} = \frac{1}{2^n}. \end{aligned}$$

Now we can prove that $E\left[W\right] = 1$, infact

$$\begin{aligned} E\left[W\right] &= E\left[\sum_{\sigma \in \Omega_n} 1_{\{\sigma \; is \; a \; P.N.E.\}}\right] \\ &= \sum_{\sigma \in \Omega_n} E\left[1_{\{\sigma \; is \; a \; P.N.E.\}}\right] \\ &= \sum_{\sigma \in \Omega_n} P(\sigma \; is \; a \; P.N.E) \\ &= 2^n P(\sigma \; is \; a \; P.N.E.) = 2^n \frac{1}{2^n} = 1 \end{aligned}$$

□

Definition 0.2. *Let f, h:* $\aleph \to \Re$, *and write* $||h|| := sup_{k>0}|h(k)|$. *We denote the total variation distance between the distributions of two random variables* W *and* Z *by*

$$\|\ell(W) - \ell(Z)\| := sup_{||h||=1} \left|E\left[h(W)\right] - E\left[h(Z)\right]\right|.$$

The goal of this work is to prove that the distribution of the number of PNE is approximately Poisson for n large, in the sense of total variation norm definited above.

This result will be obtained by the so-called Chen Stein Method for Poisson convergence [2]. A convenient form of this method can be found in the work of Arratia et al. [1].

In Chapther(2) we prove the following theorem.

Theorem 0.1. *In the independent model let* $W = \sum_{\sigma \in \Omega_n} X_\sigma$ *be the number of PNE, and let* Z *be a Poisson random variable with* $E[Z] = E[W] = 1$. *Then:*

$$\|\ell(W) - \ell(Z)\| \overset{n \to \infty}{\longrightarrow} 0.$$

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