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The cohomology algebras of compact Kähler manifolds and the Kodaira problem

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Introduction

The category of complex manifolds is much more rigid than that of topological or C^∞ manifolds: e.g. partitions of unity do not exist and on compact complex manifolds the only global sections of the structure sheaf are constant ones.
This allows us to appreciate a wide range of phenomena that such rigidity causes.
One of the most interesting (and studied) ones is that of variation of complex structure: suppose we are given two complex spaces $\mathcal{X}, \mathcal{B}$ and a smooth proper morphism between them

$$\pi : \mathcal{X} \to \mathcal{B}.$$

Moreover, suppose that over a fixed point $0 \in \mathcal{B}$ the fibre of $\pi, \pi^{-1}(0)$, is a compact complex manifold, X_0. In this case we will say that the analytic set $X_b = \pi^{-1}(b),\ b \in \mathcal{B}$ is a deformation of X_0. It is an easy consequence of Ehresmann's Theorem (see [V02, I, Ch. 9, Thm. 9.2]) that, if $\mathcal{B}$ is connected, all fibres over points in $\mathcal{B}$ are diffeomorphic, i.e. the family parametrized by $\mathcal{B}$ is trivial in the category of C^∞ manifolds. Hence all deformations of a compact complex manifold are diffeomorphic complex manifolds, while that does not happen in the category of complex manifolds, i.e. the deformations of a compact complex manifold are not necessarily biholomorphic. One of the most studied situations, in algebraic geometry, in which this phenomenon takes place, is that of moduli spaces [of geometric objects].

In this thesis, our point of view will be that of complex manifolds and of variations of their structures.
We will focus particularly on compact Kähler manifolds:

Definition. *A Kähler manifold, X, is a complex manifold endowed with a Kähler metric, i.e. a metric corresponding to a closed positive $(1,1)$ form ω, i.e.*

$$d\omega = 0.$$

Kähler geometry, on compact complex manifolds, provides several deep results, the most important of which is the famous

Hodge Decomposition Theorem. *Let X be a compact Kähler manifold, then we have a canonical decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{p\geq 0, q\geq 0 \\ p+q=k}} H^{p,q}(X) \cong \bigoplus_{\substack{p\geq 0, q\geq 0 \\ p+q=k}} H^q(X, \Omega^p).$$

Hodge Decomposition Theorem and its consequences (e.g. Hard Lefschetz Theorem, Hodge-Riemann bilinear relations) have the striking advantage to turn many important geometrical questions into questions involving linear algebra (in the cohomology vector spaces of a manifold). In this sense, they provide a linearization of many geometrical problems.

A first glimpse of that situation is offered by projective manifolds. Since $\mathbb{P}^N$ can be endowed with a Kähler metric, namely the Fubini-Study metric, and Kähler metric are well-behaved under pull-back, we see that any analytic (equivalently algebraic, by Chow's Lemma) submanifold of $\mathbb{P}^N$ is automatically Kähler. An interesting problem is to understand under what hypotheses we can carachterize projective manifolds among Kähler ones. In 1954, Kodaira solved this question with a very nice and powerful criterion:

Kodaira Theorem ([Kd54]). *A compact Kähler manifold X is projective if and only if there is a Kähler class $\beta \in H^2(X, \mathbb{Q})$, i.e. there exists a Kähler metric h, whose associated positive $(1,1)$ form ω give rise to a cohomology class $[\omega] = \beta$ contained in the rational cohomology of X.*

In view of this criterion, we see that Kähler geometry is an extension of projective geometry obtained by relaxing the rationality condition on a Kähler class.

This point of view, together with the many restrictive conditions on the topology of Kähler manifolds provided by Hodge theory (the strongest one being the formality theorem, see [DGMS75]), would indicate that compact Kähler manifolds and complex projective ones cannot be distinguished by topological invariants.
This hypothesis is supported by a result obtained by Kodaira in 1960, in the case of Kähler surfaces:as an outcome of his classification of compact complex surfaces (see [Kd63]) he found that

Theorem. *Let X be a compact Kähler surface, then X has a deformation which is projective.*

He actually found a slightly stronger result, asserting that it is possible to find a family of deformations for X such that projective manifolds are arbitrarily near to X.

The last theorem tells us that, in dimension 2, Kähler geometry on compact manifolds can be obtained by slightly varying the complex structure of projective manifolds (which is equivalent by variations of Hodge structures to the assertion made above on the relaxation of the rationality of the Kähler class). The case of dimension 1 is trivial, since all compact Riemann surfaces are projective.
Thus, we have the following

Question. *Is any compact n-dimensional Kähler manifold, $n \geq 3$, deformation equivalent to a projective manifold?*

This question is called the Kodaira problem. It is the natural higher-dimensional generalization of the case of surfaces, solved by Kodaira.

We know that for certain classes of compact Kähler manifolds, such as complex tori, the answer to the Kodaira problem is affermative. And in many examples, that naturally arise in geometry, this is tipically the situation (again moduli spaces).

The first result presented in this thesis is that the answer to the previous question is negative. We show that

Theorem ([V03])**.** *For any $n \geq 4$, there exist $n-$dimensional compact Kähler manifolds which are not deformation equivalent to any projective manifold.*

This is an immediate consequence of a stronger result obtained by Voisin

Theorem ([V03])**.** *Given $n \geq 4$, it is always possible to find n-dimensional compact Kähler manifoldswhich do not have the homotopy type of a complex projective manifold.*

In view of what we said in the first paragraph, this prevents our examples to be deformations equivalent to projective manifolds, otherwise, by Eheresmann's Theorem, they should be diffeomorphic to a projective manifold and hence they should also have the same homotopy type of a projective manifold.

To construct counterexamples, we start by a complex torus T admitting a particular endomorphism ϕ. The existence of such an endomorphism implies that T is not an abelian variety and, furthermore, its algebraic dimension (i.e. the trascendence degree of the field of meromorphic functions of T) is 0. Nonethelesse, T does not contain positive dimensional proper analytic subvarieties.
We then obtain a compact manifold X by blowing-up certain submanifolds of $T \times T$. We show the following

Theorem. *Let Y be a Kähler manifold such that there is an isomorphism of graded vector spaces*

$$\gamma : H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$$

preserving the cup-product. The Y is not a projective manifold.

We also show that, modifying the approach in the proof of this theorem, it is possible to provide new examples for which the isomorphism γ is now defined only over $\mathbb{Q}$ or only over $\mathbb{C}$ (i.e. under milder hypotheses), respectively. This is possible thanks to a lemma due to Deligne, which tells us how we can find sub-Hodge structures in cohomology starting from subsets of the cohomology ring obtained as zeros of polynomials for cup-product multiplication.

Lemma (Deligne)**.** *Let Z be as above, i.e. $Z := \{z \in H^k(X, \mathbb{C}) \mid f_i(z) = 0, i \in I\}$, for certain homogenous polynomials $f_i \in H^*(X, \mathbb{C})[X]$, where the grading is the one induced by the cohomology degree and the multiplicative structure is given by the cup-product.*
Let Z_1 be an irreducible component of Z. Assume the $\mathbb{C}$-vector space $< Z_1 >$ generated by Z_1 is defined over $\mathbb{Q}$, i.e. $< Z_1 >= B^k_{\mathbb{Q}} \otimes \mathbb{C}$, for some $B^k_{\mathbb{Q}} \subset A^k_{\mathbb{Q}}$. Then $B^k_{\mathbb{Q}}$ is a rational sub-Hodge structure of $A^k_{\mathbb{Q}}$.

With that trick and an easy application of the Hodge Index Theorem, we also establish simply connected examples, which are desingularizations of quotients of complex tori under suitable involutions (so-called generalized Kummer manifolds).

In the last part of the thesis, we analyze a further generalization of the Kodaira problem.

Question. *Let X be a compact Kähler manifold. Does there exist a bimeromorphic model X' of X (i.e. a compact complex manifold X' and a bimeromorphic map $\psi : X' \dashrightarrow X$) which deforms to a complex projective manifold?*

This will be the birational (or bimeromorphic) Kodaira problem.
Again such a generalization is quite natural: in fact, counterxamples obtained for the Kodaira problem are bimeromorphic to products of complex tori or to generalized Kummer manifolds and these are well-kown to be deformation equivalent to projective manifolds.

We will answer the birational Kodaira problem negatively, too. In fact, we shall show the following

Theorem ([V05])**.** *In any even dimension $n \geq 10$, there are Kähler compact manifolds providing counterexamples to the birational Kodaira problem, i.e. such that any of their bimeromorphic model does not have the homotopy type of a projective manifold.*

The strategy will be similar to the one used in the construction of counterexamples to the Kodaira problem.
We begin with the same torus T as above, endowed with the endomorphism ϕ. On the product $T \times \hat{T}$, where $\hat{T}$ is the dual torus of T, we take the fibre product

$$R := \mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi),$$

where E, E_ϕ are fixed rank 2 vector bundles. Then, we take X, a desingularization of the quotient of R by the action of a finite group generated by involutions.
We analyze, in details, the cohomology structure of manifolds bimeromorphic to this desingularization and we snow that

Theorem. *Let X' be any compact complex manifold bimeromorphically equivalent to X, and let Y be a Kähler compact manifold. Assume there is an isomorphism of graded algebras:*

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X', \mathbb{Q}).$$

Then Y is not projective.

Also in this case, the proof will strongly rely on Deligne's Lemma and the same Hodge Index argument already used. More precisely we shall show that, as a consequence of the hypotheses on X', the existence of γ imply that there are not rational polarizations on the cohomology of Y.

Note that, since any X' is bimeromorphic to a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over a quotient of tori,

the Kodaira dimension of X' is $-\infty$.
After these counterexamples, we can identify the following **open problems**, which will not be treated in this thesis:

- what is the answer to the Kodaira problem in dimension 3?
- is the birational Kodaira problem true in the case of manifolds whose Kodaira dimension is non-negative?

The first is the only open case remained in the Kodaira problem but both of these questions are really interesting for their connections with the classification of compact complex manifolds. Particularly, the latter (see Section 1.5.4) would provide an extremely surprising result. Indeed, it would show that a property, defined only in terms of the algebraic (or analytic) structure of a manifold, the Kodaira dimension, imposes a strong topological condition, i.e. that homotopy type of manifolds whose Kodaira dimension is non-negative is completely determined by projective ones.

Contents

Chapter 1

Preliminaries

1.1 Kähler manifolds

On a complex vector space V, a Hermitian bilinear form h is decomposed into real and imaginary parts as $h = g - i\omega$, where g is a symmetric real bilinear form and ω is a real 2-form which is of type $(1,1)$ for the complex structure on V. Here the notion of (complex valued) form of type (p,q) on V is the following: the space $V^* \otimes C$ of complex valued forms on V splits as a direct sum of $(V^*)^{1,0} \oplus (V^*)^{0,1}$, where $(V^*)^{1,0}$ is the space of $\mathbb{C}$-linear forms and $(V^*)^{0,1}$ is its complex conjugate. Then forms of type (p,q) are generated by forms $\alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta_1 \wedge \cdots \wedge \beta_q$, $\alpha_i \in (V^*)^{1,0}$, $\beta_j \in (V^*)^{0,1}$. The correspondence $h \to \omega$ is a bijection between Hermitian bilinear forms and real forms of type $(1,1)$ on V . Thus the notion of (semi)-positivity for Hermitian bilinear forms provides a corresponding notion of (semi)-positivity for real forms of type (1, 1). Note that when h is positive definite, ω is non degenerate, i.e. $\omega^n \neq 0$, $n = \dim_{\mathbb{C}} V$.
On a complex manifold X, the tangent space TX_x at any point is endowed with a complex structure, and the above correspondence induces a bijective correspondence between Hermitian bilinear forms on TX, and real 2-forms of type $(1,1)$ on X, i.e. of type $(1,1)$ on TX_x for any $x \in X$. In particular, if h is a Hermitian metric on TX, one can write $h = g - i\omega$, where g is a Riemannian metric (compatible with the complex structure), and ω is a positive real $(1,1)$-form.

Definition 1.1.1. *The metric h is said to be Kähler if furthermore the 2-form ω is closed.*

We can restate that as follows: a complex manifold is Kähler if it admits a Hermitian metric, written in local holomorphic coordinates as

$$h = \sum_{i,j} h_{ij} dz_i \otimes d\bar{z}_j,$$

satisfying the property that the corresponding real $(1,1)$-form

$$\omega := \frac{i}{2} \sum_{i,j} h_{i,j} dz_i \wedge d\bar{z}_j$$

is closed.
There are a number of other local characterizations of such metrics. The most useful one is the fact that at each point, there are holomorphic local coordinates centered at this point, such that the metric can be written as

$$h = \sum_{i,j} dz_i \otimes d\bar{z}_j + O(|z|^2).$$

1.1.1 Laplacians

Let Y be a compact differentiable manifold equipped with a metric g. We then have an induced metric $(,)$ on each vector bundle $\Omega^k_{X,\mathbb{R}}$; if $e_1, \dots, e_n$ is an orthonormal basis for $(T_{X,x}, g_x)$ and e_i^* is the dual basis, the $e_{i_1}^* \wedge \dots \wedge e_{i_k}^*$ form an orthonormal basis for the metric $(,)_x$ on $\Omega^k_{X,x}$.
Assume now that X is compact and oriented, and let Vol be the volume form of X relative to g. The L^2 metric on the space $\mathcal{E}^k(X)$ of differential forms on X is defined by

$$(\alpha, \beta)_{L^2} = \int_X (\alpha, \beta) Vol,$$

where (α, β) is the function $x \mapsto (\alpha_x, \beta_x)$ on X, which is continuous whenever α, β and g are continuous.
Let $n = \dim_{\mathbb{R}} X$. For each $x \in X$, we have a natural isomorphism, given by the proper exterior product

$$p : \bigwedge^{n-k} \Omega_{X.x} \to Hom(\bigwedge^{k} \Omega_{X,x}, \bigwedge^{n} \Omega_{X,x}).$$

The metric $(,)_x$ gives an isomorphism

$$m : \bigwedge^{k} \Omega_{X,x} \to Hom(\bigwedge^{k} \Omega_{X,x}, \mathbb{R}),$$

thanks to the orientability assumption. We can thus define the operator

$$*_x = p^{-1} \circ m : \bigwedge^{k} \Omega_{X,x} \to \bigwedge^{n-k} \Omega_{X,x},$$

which varies differentiably with x when g is differentiable, and which is of the same class as g.

Definition 1.1.2. *Let $*$ denote the isomorphism of vector bundles*

$$* : \Omega^k_{X,\mathbb{R}} \to \Omega^{n-k}_{X,\mathbb{R}}$$

constructed above. Let $*$ also denote the induced morphism on the level of sections, i.e. of differential forms

$$* : \mathcal{E}^k(X) \to \mathcal{E}^{n-k}(X).$$

The operator $*$ is called the Hodge operator.

The essential property of $*$ is the following

$$\forall \alpha, \beta \in \mathcal{E}^k(X),\ (\alpha, \beta)_{L^2} = \int_X \alpha \wedge *\beta \tag{1.1}$$

One has the Laplacian Δ_d acting on differential forms, preserving the degree, defined as

$$\Delta_d := dd^* + d^*d,$$

where $d^* := \pm * d*$ is the formal adjoint of d.
In the case of a compact complex manifold X, equipped with a hermitian metric h, the g above will be given by the real part of h that descends to a metric on the real tangent bundle of X.
Hodge theory states that if X is compact and orientable, any de Rham cohomology class has a unique representative α which is a C^∞ form both closed and coclosed (i.e. $d\alpha = d(*\alpha) = 0$), or equivalently harmonic (i.e. $\Delta_d \alpha = 0$).
On the other hand, on a general complex manifold endowed with a Hermitian metric, the operator d splits as

$$d = \partial + \bar{\partial},$$

where each operator preserves, up to a shift, the bigrading given by decomposition of C^∞-forms into forms of type (p, q). We can also extend by $\mathbb{C}$-linearity the operator $*$ on complex forms on X.
Moreover, it is always possible to associate to ∂ and $\bar{\partial}$ corresponding Laplacians Δ_∂ and $\Delta_{\bar{\partial}}$ defined as

$$\Delta_\partial := \partial\partial^* + \partial^*\partial,\ \Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}\bar{\partial},$$

where the $*$ means the formal adjoint for the induced metric on forms. For obvious formal reasons Δ_∂ and $\Delta_{\bar{\partial}}$ preserve the bigrading and even the bidegree. However, it is not the case in general that Δ_d preserves the bigrading. It turns out that, as a consequence of the so-called Kähler identities, when the metric h is Kähler, one has the relation

$$\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}},$$

which implies that Δ_d preserves the bidegree.
This has for immediate consequence the fact that each cohomology class can be written as a sum of cohomology classes of type (p, q), where cohomology classes of type (p, q) are defined as those which can be represented by closed forms of type (p, q).

1.1.2 Hodge Theorem

The main consequence of Hodge theory can be stated as follows:

Theorem 1.1.3. *Let X be a compact Kähler manifold. Then there is a canonical decomposition*

$$H^k(X, \mathbb{C}) = \bigoplus_{\substack{p+q=k,\\ p\geq 0, q\geq 0}} H^{p,q}(X),$$

where the $H^{p,q}$ are defined as above.
Furthermore we can identify $H^{p,q} = H^q(X, \Omega^p)$. The $H^{p,q}$'s satisfy a further property called Hodge symmetry:

$$\forall k, \forall p, q \text{ s.t. } p + q = k,\ H^{p,q} = \overline{H^{q,p}}.$$

The main benefit of Hodge theory is the fact that it allows us to translate many geometrical problems into linear algebra problems relating to the cohomology ring of a variety. This principle will be probably one of the most relevant aspects of this thesis, even if not the central one.

Example 1.1.4. Not all complex manifolds are Kähler manifolds.
A classical example of a complex manifold which is not Kähler is provided by the so-called Hopf surfaces: let $A \in GL(2, \mathbb{C})$ be a matrix with eigenvalues of norm > 1 and let $< A > \subset GL(2, \mathbb{C})$ be the subgroup generated by A. Clearly $< A > \cong \mathbb{Z}$. The action of A on $X = \mathbb{C}^2 \setminus \{0\}$ is free and properly discontinuous. Therefore the quotient $S_A = X/A$ is a compact complex manifold called Hopf surface: the holomorphic map $X \to S_A$ is the universal cover and then for every point $x \in S_A$ there exists a natural isomorphism $\pi_1(S_A, x) \cong \mathbb{Z}$. This implies that S_A is not Kähler, otherwise $\pi_1(S_A, x)$ wuold have even rank, by Hodge symmetry.
It is a well known result, instead, that any analytic submanifold (or any algebraic submanifold, by Chow's Theorem) of $\mathbb{P}^n$ is Kähler. Indeed, on $\mathbb{P}^n$, we have a well-defined Kähler metric, called the Fubini-Study metric (see [V02, I, Ch. 3, § 3.3.2]). By pulling-back the Fubini-Study metric on the given submanifold of $\mathbb{P}^n$, we obtain a Kähler metric. As we will see below, among Kähler manifolds, projective ones are carachterized by a special cohomological condition, which was discovered by Kodaira in [Kd54].

Theorem (1.1.3) has a wide number of consequences and applications (e.g. see [V02], [GH78], [Le91]). We will not state them in this section, but rather we will recall them when needed throughout the thesis.

Example 1.1.5. We will work out the Hodge decomposition in a special example that will be of fundamental importance in the next chapters.
Let X be an n-dimensional complex torus, i.e. the quotient of $\mathbb{C}^n$ by a $2n$-dimensional lattice $\Lambda \subset \mathbb{C}^n$. We have a quotient map

$$\pi : \mathbb{C}^n \to \mathbb{C}^n/\Lambda$$

which is also the universal covering map. Furthermore, X is also a complex commutative Lie group: as a consequence, we have that the tangent bundle TX is trivial and given linear coordinates $z_1, \dots, z_n$ on $\mathbb{C}^n$, the images of the vector fields $\frac{\partial}{\partial z_i}, i = 1, \dots, n$ via the map $d\pi_* : T\mathbb{C}^n \to TX$ give n linearly indipendent sections of TX, hence a trivialization. The same happens for the standard Hermitian metric on $T\mathbb{C}^n$:

$$H = \sum_i dz_i \otimes d\bar{z}_i.$$

The real part of H, G is the standard Euclidean metric on the underlying real vector space. If we take real linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ by the equalities

$$z_j = x_i + iy_j, j = 1, \ldots, n,$$

G is simply described by the formula

$$G = \sum_i dx_i^2 + dy_i^2.$$

It is immediate to show that the metric H induces a Hermitian metric H_X on the tangent bundle TX, which is Kähler (since it is Kähler on $\mathbb{C}^n$).
From a cohomological point of view, it is a well known fact that X is homeomorphic to the cartesian product of $2n$ copies of S^1. It is a straightforward application of the cup-product properties that there is a isomorphism

$$H^*(X, \mathbb{Z}) \cong \bigwedge^* H^1(X, \mathbb{Z})$$

and by universal coefficient theorem the same is true with rational (real, complex) coefficients.
By definition the Laplacian induced by G on differential forms of degree k on $\mathbb{C}^n$ is nothing but the usual Laplacian, computed on the coefficients, i.e.

$$\Delta_d \alpha = \Delta_d(\sum_{\substack{I,J \\ |I|+|J|=k}} \alpha_{IJ} dx_I \wedge dy_J) = \sum_{\substack{I,J \\ |I|+|J|=k}} \Delta \alpha_{IJ} dx_I \wedge dy_J,$$

where the α_{IJ} are C^∞ complex functions on $\mathbb{C}^N$. Hence a differential form of degree k is harmonic if and only if its coefficients are harmonic functions.
Since by the maximum principle, harmonic functions on the torus are costants, we deduce that harmonic forms on X are induced by differential forms with costant coefficients on $\mathbb{C}^n$. These have the form

$$\sum_{\substack{I,J \\ |I|+|J|=k}} c_{IJ} dx_I \wedge dy_J, c_{IJ} \in \mathbb{C}. \tag{1.2}$$

Since $z_j = x_j + iy_j$, it follows that

$$dz_j = dx_j + idy_j, \ d\bar{z}_j = dx_j + idy_j, \ j = 1, \ldots, n.$$

Thus, we can write (1.2) in the form

$$\sum_{\substack{p \geq 0, q \geq 0 \\ p+q=k}} \sum_{\substack{I,J \\ |I|=p, |J|=q}} s_{IJ} dz_I \wedge d\bar{z}_J, s_{IJ} \in \mathbb{C}$$

which naturally gives the Hodge decomposition on the k-th cohomology group with complex coefficients.

1.2 Hodge structures

We have already said that the complex valued de Rham algebra $\mathcal{E}^*(X)$ of a complex manifold splits as a direct sum $\mathcal{E}^k(X) = \bigoplus_{p+q=k} \mathcal{E}^{p,q}(X)$, where $\mathcal{E}^{p,q}(X)$ is the set of C^∞ differential forms of type (p,q). The Hodge decomposition theorem says that, when X is compact Kähler, this decomposition descends to the cohomology groups with complex coefficients.
On the other hand, cohomology groups with complex coefficients are complex vector space endowed with a canonical integral structure, given by the universal coefficients theorem:

$$H^k(X,\mathbb{C}) = H^k(X,\mathbb{Z}) \otimes \mathbb{C}.$$

In the sequel we will denote by $H^k(X,\mathbb{Z})$, the integral cohomology of X modulo torsion. Thus, $H^k(X,\mathbb{Z})$ is a lattice in $H^k(X,\mathbb{C})$, and Hodge theory provides us with an interesting continuous invariant attached to a Kählerian complex structure on X, namely the position of the complex subspaces $H^{p,q}(X)$, $p+q=k$ with respect to the lattice $H^k(X,\mathbb{Z})$. This leads to the notion of Hodge structure.

Definition 1.2.1. *A weight k (integral) Hodge structure is a lattice V, with a decomposition*

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}, \quad \overline{V^{q,p}} = V^{p,q},$$

where $V_{\mathbb{C}} := V \otimes \mathbb{C}$.
By taking a rational space instead of a lattice, we can define analogously a rational Hodge structure.

The contents of the previous section can be summarized by saying that if X is a compact Kähler manifold, each cohomology group (modulo torsion) $H^k(X,Z)$ carries a canonical Hodge structure of weight k.
Given a weight k Hodge structure V, we can define a representation ρ of $\mathbb{C}^*$ on $V_{\mathbb{R}}$, defined by the condition that $z \in \mathbb{C}^*$ acts by multiplication by $z^p \bar{z}^q$ on $V^{p,q}$. Then the restriction of ρ to $\mathbb{R}^*$ is the map $\mathbb{R} \ni \lambda \mapsto \lambda^k Id \in Hom(V_{\mathbb{C}}, V_{\mathbb{C}})$. Conversely, given a representation of $\mathbb{C}^*$ on $V_{\mathbb{R}}$ satisfying the last condition, the associated character decomposition of $V_{\mathbb{C}}$ will provide a Hodge structure on V (see [V02, I, Ch. 6, page 154]).
Given a Hodge structure (defined over $\mathbb{Z}$ or over $\mathbb{Q}$) H of weight h, in the sequel we will sometimes consider the so-called Hodge filtration defined by

$$F^l H_{\mathbb{C}} = \bigoplus_{k \geq l} H^{k,h-k}, \ H_{\mathbb{C}} := H \otimes \mathbb{C}.$$

Notice that the Hodge filtration of H satisfies the following two properties:

1. $\forall p \in \mathbb{Z},\ F^p H_{\mathbb{C}} \cap \overline{F^{h-p+1} H_{\mathbb{C}}} = \emptyset$;
2. $\forall p \in \mathbb{Z},\ F^p H_{\mathbb{C}} \oplus \overline{F^{h-p+1} H_{\mathbb{C}}} = H_{\mathbb{C}}$.

The data of a Hodge filtration $F^pH_{\mathbb{C}}, p \in \mathbb{Z}$ satisfying the two properties above, is equivalent to the data of the Hodge decomposition, because by Hodge symmetry, we recover $H^{p,q}$ as

$$H^{p,q} = F^pH_{\mathbb{C}} \cap \overline{F^qH_{\mathbb{C}}}.$$

In view of the definition of the Hodge filtration $F^pH_{\mathbb{C}}, p \in \mathbb{Z}$ of a Hodge structure H, we will also indicate H by the notation

$$(H, F^pH_{\mathbb{C}}).$$

A number of operations can be done in the category of Hodge structures. We can take the direct sum of two Hodge structures of weight k: the lattice is the direct sum of the two lattices, and the (p, q) components are the direct sum of the (p, q)-components of each term. We can take the dual of a Hodge structure of weight k, which will have weight $-k$. Its underlying lattice is the dual of the original one, and its Hodge decomposition is dual to the original one, with the rule $(V^{p,q})^* = (V^*)^{-p,-q}$. With this definition, we can verify that if X is a compact Kähler manifold of dimension n, whose cohomology has no torsion, the Hodge structures on $H^k(X, \mathbb{Z})$ and $H^{2n-k}(X, \mathbb{Z})$ are dual via Poincaré duality, up to a shift of the degree. The tensor product of two Hodge structures V, W of weight k, l is the Hodge structure of weight $k + l$ whose underlying lattice is $M = V \otimes W$ and which has

$$M^{p,q} = \bigoplus_{r+t=p,s+u=q} V^{r,s} \otimes W^{t,u}.$$

1.2.1 Morphisms of Hodge structure

Definition 1.2.2. *A morphism of Hodge structure $V = \oplus_{p+q=k}V^{p,q}, W = \oplus_{s+t=k+2r}W^{p,q}$ of respective weights $k, k+2r$ is a morphism of lattices $\phi : V \to W$, such that the $\mathbb{C}$-linear extension $\phi_{\mathbb{C}}$ of ϕ sends $V^{p,q}$ to $W^{p+r,q+r}$. Such a morphism is said to be of bidegree (r, r), as it shifts by (r, r) the bigrading given by the Hodge decomposition.*

Natural examples of morphisms of Hodge structure are induced by holomorphic maps $f : X \to Y$ between compact Kähler manifolds. The pull-back map in cohomology

$$f^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$$

is a morphism of Hodge structure of weight k. Indeed, the pull-back by f of a closed form of type (p, q) is again a closed form of type (p, q).
We also have the Gysin map

$$f_* : H^k(X, \mathbb{Z}) \to H^{k+2r}(Y, \mathbb{Z}),\ r := \dim_{\mathbb{C}} Y - \dim_{\mathbb{C}} X.$$

It is defined on integral cohomology as the composition

$$PD_Y^{-1} \circ f^* \circ PD_X,$$

where PD_X is the Poincaré duality isomorphism

$$PD_X : H^l(X, \mathbb{Z}) \to H_{2n-l}(X, \mathbb{Z}), n = \dim_{\mathbb{C}} X$$

and similarly for PD_Y, while the map f_* at the middle is the natural push-forward map induced on homology by f. We can also define the Gysin morphism (on the cohomology modulo torsion) as the composition

$$H^k(X, \mathbb{Z}) \xrightarrow{PD} H^{2n-k}(X, \mathbb{Z})^* \xrightarrow{(\phi^*)^t} H^{2n-k}(Y, \mathbb{Z})^* \xrightarrow{PD} H^{k+2r}(Y, \mathbb{Z}),$$

where PD is the Poincare duality morphism induced in cohomology by the perfect pairing

$$H^k(X, \mathbb{Z}) \otimes H^{2n-k}(X, \mathbb{Z}) \to H^{2n}(X, \mathbb{Z}) \cong \mathbb{Z},$$

and $(\phi^*)^t$ is the adjoint of ϕ^*. One can easily prove that f_* is a morphism of Hodge structure of bidegree (r, r).
Up to now, we have been working with integral Hodge structures. It is sometimes more convenient to use rational Hodge structures. Morphisms of rational Hodge structures are defined in the same way as above, morphisms of lattices being replaced with morphisms of $\mathbb{Q}$-vector spaces. Given a morphism of Hodge structure $\phi : V_{\mathbb{Q}} \to W_{\mathbb{Q}}$ there is an obvious induced Hodge structure on $\ker \phi$, due to the fact that, since ϕ preserves up to a shift the bigrading given by Hodge decomposition, its kernel is stable under Hodge decomposition. For the same reason there is an induced Hodge structure on $Coker\phi$. Thus we have kernels and cokernels in the category of rational Hodge structures.
Let us note the following important fact, that we will use very often in the next chapters.

Proposition 1.2.3. *Let $\phi : X \to Y$ be a holomorphic surjective map, between compact complex manifolds, with X Kähler. Then the map $\phi^* : H^k(Y, \mathbb{Q}) \to H^k(X, \mathbb{Q})$ is injective for every k.*

Proof. [V02, I, Ch. 7, Lemma 7.28] □

Remark 1.2.4. The rational coefficients in the previous proposition can be replaced by integral (in this case we take torsion free cohomology), real or complex coefficients.

Remark 1.2.5. When $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X$, we do not need the Kähler hypothesis on X. When the morphism ϕ has finite generic fibre of cardinality d, it is of degree $d > 0$, since its differential is $\mathbb{C}$-linear and thus preserves the orientation. In this case we have the following formula:

$$\phi_* \circ \phi^* : H^k(Y, \mathbb{Z}) \to H^k(Y, \mathbb{Z}), \;\; \phi_* \circ \phi^* = d \cdot Id_Y$$

In particular, we find that if $\phi : X \to Y$ is a differentiable map of degree $d \neq 0$ between compact differentiable manifolds, the map $\phi^* : H^k(Y, \mathbb{Z}) \to H^k(X, \mathbb{Z})$ is injective for every k.

Another simple application of the definition of morphism of Hodge structure is the following: given a cohomology class $\alpha \in H^{2r}(X,\mathbb{Z}) \cap H^{r,r}(X)$ on a compact Kähler manifolds X, this gives rise to morphisms of Hodge structure

$$\cup\alpha : H^k(X,\mathbb{Z}) \to H^{k+2r}(X,\mathbb{Z}), \forall k \in \mathbb{N}$$

a fact which will be very much used in the sequel. It is immediate to see that an analoguos statement is true when $\alpha \in H^{2r}(X,\mathbb{Q}) \cap H^{r,r}(X)$ and the map induced by cup-product multiplication by α is in rational cohomology.

1.2.2 Hodge classes

Let X be compact Kähler; we have the Hodge decomposition on $H^{2k}(X,\mathbb{C})$. We make the following definitions:

Definition 1.2.6. *An integral Hodge class of degree $2k$ on X is a class in $H^{2k}(X,\mathbb{Z})$ whose image in $H^{2k}(X,\mathbb{C})$ is of type (k,k). We will denote the group of such classes by $Hdg^{2k}(X,\mathbb{Z})$. One defines similarly the space of rational Hodge classes by $Hdg^{2k}(X) := H^{2k}(X,\mathbb{Q}) \cap H^{k,k}(X)$.*

Here are some classic examples of Hodge classes.

- Cohomology classes of closed analytic subspaces $Z \subset X$ of codimension k are of Hodge type (k,k). Indeed, the singular locus Z_{sing} of such a Z is then a closed analytic subset of X which has codimension $\geq k+1$ and thus real codimension $\geq 2k+2$. Thus one can define $[Z] \in H^{2k}(X,\mathbb{C})$ by taking the cohomology class $[Z] \in H^{2k}(X \setminus Z_{sing},\mathbb{Z})$ of the closed complex submanifold $Z \setminus Z_{sing} \subset X \setminus Z_{sing}$ and by observing that $H^{2k}(X \setminus Z_{sing},\mathbb{Z}) \cong H^{2k}(X,\mathbb{Z})$. The class $[Z]$ is an integral Hodge class. This can be seen using Lelong's theorem, showing that the current of integration over $Z \setminus Z_{sing}$ is well-defined and closed, with cohomology class equal to the image of $[Z]$ in $H^{2k}(X,\mathbb{C})$.On the other hand, this current annihilates all forms of type (p,q), $p \neq q$, $p+q = 2n-2k$, $n = \dim_{\mathbb{C}} X$, and it follows dually that its class is of type (k,k). Let us recall the famous Hodge conjecture, which is intimately related to classes of analytic proper subvarieties of a given projective manifold:

  Hodge Conjecture. *Let X be a complex projective manifold. Then the space $Hdg^{2k}(X)$ of degree $2k$ rational Hodge classes on X is generated over $\mathbb{Q}$ by classes of the form $[Z]$ constructed above, for suitable analytic closed subsets $Z \subset X$.*

- If E is a complex vector bundle on a topological manifold X, we have the rational Chern classes $c_i(E) \in H^{2i}(X,\mathbb{Q})$. (Note that the Chern classes are usually defined as integral cohomolgy classes, $c_i \in H^{2i}(X,\mathbb{Z})$, but in the text, the notation c_i will be used for the rational ones.) If E is now a holomorphic vector bundle on a complex manifold X, the Chern classes of E are Hodge classes. This follows indeed from Chern–Weil theory, which provides de Rham representatives of $c_i(E)$ as follows:

if ∇ is a complex connection on E, with curvature operator $R_\nabla \in \mathcal{E}_X^2 \otimes End(E)$, then a representative of $c_k(E)$ is given by the degree $2k$ closed form $\sigma_k(\frac{i}{2\pi}R_\nabla)$, where σ_k is the polynomial invariant under conjugation on the space of matrices, which to a matrix associates the k-th symmetric function of its eigenvalues. Now, if E is a holomorphic vector bundle on E, there exists a complex connection ∇ on E such that R_∇ is of type $(1,1)$, i.e. $R_\nabla \in \mathcal{E}_X^{1,1} \otimes End(E)$. Given a Hermitian metric h on E, one can take the so-called Chern connection, which is compatible with h, and has the property that its $(0,1)$-part is equal to the $\bar{\partial}$ operator of E. This implies that $\sigma_k(\frac{i}{2\pi}R_\nabla)$, and shows that $c_k(E)$ is Hodge.

Remark 1.2.7. In the case of a compact Kähler manifold X which is also projective, we have a strong result asserting that these two construction generates the same subgroups of $Hdg^{2k}(X)$:

Theorem 1.2.8. *Let X be an algebraic variety. Then the subgroup of $Hdg^{2k}(X)$ generated by cohomology classes of analytic subsets of X, and the subgroup generated by Chern classes of holomorphic vector bundles, coincide.*

We have the cup-product between the cohomology groups $H^k(X,\mathbb{Z})$ of a manifold (topological space would be enough):

$$\cup : H^k(X,\mathbb{Z}) \otimes H^l(X,\mathbb{Z}) \to H^{k+l}(X,\mathbb{Z}). \tag{1.3}$$

At the level of complex cohomology, where cohomology classes are represented via de Rham theory as classes of closed forms modulo exact ones, the cup-product is given by the exterior product, namely, if $\alpha \in H^k(X,\mathbb{C}), \beta \in H^l(X,\mathbb{C})$ are represented respectively by closed complex valued differential forms $\tilde{\alpha}, \tilde{\beta}$, then $\alpha \cup \beta$ is represented by the closed differential form $\tilde{\alpha} \wedge \tilde{\beta}$. Now, if X is a complex manifold and $\tilde{\alpha}, \tilde{\beta}$ are respectively of type $(r,s), r+s=k$, $(t,u), t+u=l$, then $\tilde{\alpha} \wedge \tilde{\beta}$ is closed of type $(r+t, s+u)$. Thus, if X is a compact complex manifold, the definition of the $H^{p,q}$ groups of X shows that $H^{r,s}(X) \cup H^{t,u} \to H^{r+s,t+u}$. Using the definition of the Hodge structure on the tensor product $H^k(X,\mathbb{Z}) \otimes H^k(X,\mathbb{Z})$ this amounts to say that the cup-product is a morphism of Hodge structure of weight $k+l$. Moreover, Poincaré duality and cup product allows us to determine pairings of the form

$$H^k(X,\mathbb{Z}) \otimes H^{2n-k}(X,\mathbb{Z}) \to H^{2n}(X,\mathbb{Z}),\ n = \dim_{\mathbb{C}} X.$$

Poincaré duality simply tells us that these pairings are perfect, i.e. gives a duality of vector spaces between $H^k(X,\mathbb{Z})$ and $H^{2n-k}(X,\mathbb{Z})$, once we have choosen an isomorphism $H^{2n}(X,\mathbb{Z}) \cong \mathbb{Z}$. Again using the definition of the subgroups $H^{p,q}(X)$, the following result can be easily proved.

Proposition 1.2.9. *We have*

$$H^{r,s}(X) = \Big\{ \bigoplus_{\substack{t+v=2n-r-s \\ (t,v)\neq(n-r,n-s)}} H^{t,v}(X) \Big\}^{\perp},$$

where the ortogonality is relative to the Poincaré duality on Y

Proof. See [V02, Ch. 7, Lemma 7.30] □

1.3 Polarizations

A very deep application of Hodge theory is the hard Lefschetz theorem, which says the following: let X be a compact Kähler manifold of complex dimension n and $\omega \in H^2(X,\mathbb{R})$ the class of a Kähler form Ω on X. Cup-product with ω gives an operator usually denoted by $L : H^*(X,\mathbb{R}) \to H^{*+2}(X,\mathbb{R})$. We will denote by L^j the composition $L \circ \cdots \circ L$, j times. Clearly, $L^j : H^*(X,\mathbb{R}) \to H^{*+2j}(X,\mathbb{R})$. By abuse of notation, we will often indicate by L^j its restriction to $H^l(X,\mathbb{R}) \subset H^*(X,\mathbb{R})$, too.

Theorem 1.3.1 (**Hard Lefschetz Theorem**). *For any $k \leq n$, the restriction of the the map L^{n-k} to $H^k(X,\mathbb{R})$ gives an isomorphism*

$$L^{n-k}_{|H^k(X,\mathbb{R})} : H^k(X,\mathbb{R}) \to H^{2n-k}(X,\mathbb{R}).$$

The proof involves first a pointwise computation, saying that wedge product with Ω^{n-k} induces a pointwise isomorphism between the k-th and $2n-k$-th exterior powers of the complexified cotangent bundle of X. The second ingredient is the fact that wedge product with the Kähler form Ω preserves harmonic forms, which are the canonical de Rham representatives of cohomology classes on X, once we have given the Kähler metric. Thus one has to check the result by looking at the wedge product with Ω^{n-k} on harmonic forms. And the last ingredient is Poincaré duality, which says that both spaces have the same dimension, so that bijectivity is equivalent to injectivity.
It is interesting to note that if X is projective, then we can take for ω the first Chern class of a very ample line bundle (i.e. a line bundle providing a projective immersion of X, see [GH78]) and then the hard Lefschetz theorem (see [V02, II, Ch. 1-2]) implies immediately the injectivity statement in the Lefschetz theorem on hyperplane sections, at least for smooth hyperplane sections and rational coefficients.
A first formal consequence of the hard Lefschetz Theorem (1.3.1) is the so-called Lefschetz decomposition. With the same notations as before, let us define for $k \leq n$ the primitive cohomology of X in degree k, by

$$H^k(X,\mathbb{R})_{prim} := \ker\{L^{n-k+1} : H^k(X,\mathbb{R}) \to H^{2n-k+2}(X,\mathbb{R})\}$$

For example, if $k = 1$, the whole cohomology is primitive, and if $k = 2$, primitive cohomology is the same as the orthogonal subspace, with respect to Poincaré duality, of $\omega^{n-1} \in H^{2n-2}(X,\mathbb{R})$. The Lefschetz decomposition is the following

Theorem 1.3.2 (**Lefschetz decomposition**). *The cohomology groups $H^k(X,\mathbb{R})$, $k \leq n$ decomposes as*

$$H^k(X,\mathbb{R}) = \bigoplus_{2r \leq k} L^r H^{k-2r}(X,\mathbb{R})_{prim}$$

Note that Lefschetz decomposition can also be extended to the case $k > n$ using the hard Lefschetz isomorphism.

Remark 1.3.3. If $\omega \in H^2(X,\mathbb{Q})$, then the same things are true for the rational cohomology of X.

1.3.1 Hodge-Riemann bilinear relations

We consider a Kähler compact manifold X with Kähler class ω. We can define an intersection form q_ω on each $H^k(X,\mathbb{R})$ by the formula

$$q_\omega(\alpha,\beta) = \int_X \omega^{n-k} \cup \alpha \cup \beta.$$

By the hard Lefschetz theorem and Poincaré duality, q_ω is a non-degenerate bilinear form (i.e. its representing matrix with respect to a fixed basis of $H^k(X,\mathbb{R})$ is invertible, no matter how the basis is choosen). It is skew-symmetric if k is odd and symmetric if k is even.
It is clear that we can extend q_ω to $H^*(X,\mathbb{C}) = H^*(X,\mathbb{R})\otimes\mathbb{C}$: in this case, the extension of q_ω satisfies the property that

$$q_\omega(\alpha,\beta) = 0, \alpha \in H^{p,q},\ \beta \in H^{p',q'},\ (p',q') \neq (q,p).$$

This property is indeed an immediate consequence of the fact that $H^{2n}(X,\mathbb{C}) = H^{n,n}(X),\ n = \dim_{\mathbb{C}} X$ by Hodge symmetry, as $H^{2n}(X)$ is 1-dimensional. Another way to rephrase this is to say that the Hermitian pairing h_ω on $H^k(X,\mathbb{C})$ defined by

$$h_\omega(\alpha,\beta) = i^k q_\omega(\alpha,\bar{\beta})$$

has the property that the Hodge decomposition is orthogonal with respect to h_ω.

Remark 1.3.4. This property is summarized under the name of **first Hodge-Riemann bilinear relations**

Coming back to q_ω, note also that the Lefschetz decomposition is orthogonal with respect to it. Indeed, if $\alpha,\beta \in H^k(X,\mathbb{C})$, $\alpha = L^r\alpha'$, $\beta = L^s\beta'$, $r < s$ and α', β' are primitive, then

$$L^{n-k}\alpha \cup \beta = L^{n-k+r+s}\alpha' \cup \beta',$$

and $L^{n-k+r+s}\alpha' = 0$, since by primitivity $L^{n-k+2r+1}\alpha' = 0$.
The second Hodge-Riemann bilinear relations given below play a crucial role, especially in the study of the period maps. Note first that, because the operator L shifts the Hodge decomposition by $(1,1)$, the primitive cohomology has an induced Hodge decomposition:

$$H^k(X,\mathbb{C})_{prim} = \oplus_{p+q=k} H^{p,q}_{prim}(X),$$

where $H^{p,q}(X)_{prim} := H^{p,q}(X) \cap H^r(X,\mathbb{C})_{prim}$. We have now the following result:

Theorem 1.3.5 (**Second Hodge-Riemann bilinear relations**). *The sesquilinear form h_ω is definite of sign $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k}$ on the component $L^r H^{p,q}(X)_{prim}$, $2r+p+q=k$ of $H^k(X,\mathbb{C})$*

Example 1.3.6. A basic application of this theorem is the well known Hodge index theorem for the intersection form on 2-degree cohomology group of a compact Kähler surface X, equipped with a Kähler class $\omega \in H^2(X,\mathbb{R})$. As we are looking at the middle cohomology, the form q_ω is equal to the natural intersection pairing on $H^2(X)$. In this case, the primitive cohomology is the orthogonal complement of the Kähler form and the second Hodge-Riemann bilinear conditions say that q_ω is negative definite on the real part of $H^{1,1}(X)_{prim}$ and positive definite on the real part of $H^{2,0}(X) \oplus H^{0,2}(X)$. It is also obviously positive on the line given by ω, which is perpendicular to both of these spaces. This shows that the Hodge numbers of compact Kähler surfaces are determined by their topology, which is not the case in higher dimension.

Another application of this principle, which will be used later on, is the following

Lemma 1.3.7. *Let X be a compact Kähler manifold. Assume there is a rank 2 subspace $V \subset H^2(X, \mathbb{R})$ such that $\forall \alpha, \beta \in V$, $\alpha \cup \beta = 0 \in H^4(X, \mathbb{R})$, then the Hodge structure on $H^2(X, \mathbb{Q})$ is non trivial.*

Proof. Indeed, if $H^2(X, \mathbb{Q})$ were trivial, that is entirely of type $(1,1)$, then for ω a Kähler form on X, the non-degenerate intersection form q_ω on $H^2(X, \mathbb{R})$ would have one positive sign, and thus the dimension of a maximal isotropic subspace would be 1. But V is isotropic, which is a contradiction. □

1.4 Blow-ups

1.4.1 Vector bundles

Recall the following

Definition 1.4.1. *Let X be a complex manifold, a n-dimensional holomorphic vector bundle E over X, is a complex manifold E together with a map $\pi : E \to X$ such that $\forall x \in X$ there is a neighbourhood U of x and a biholomorphic map $\tau_U : \pi^{-1}(U) \to U \times \mathbb{C}^n$ such that the following diagram commutes*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\quad \tau_U \quad} & U \times \mathbb{C}^n \\ & {\scriptstyle \pi} \searrow \qquad \swarrow {\scriptstyle pr_1} & \\ & U & \end{array} \tag{1.4}$$

and for any $z \in U_1 \cap U_2 \subset X$, U_1, U_2 neighbourhoods for which there are trivializing maps τ_{U_1}, τ_{U_2} as in (1.4), the transition functions

$$\tau_{U_1} \circ \tau_{U_2}^{-1} : \tau_{U_2}(U_1 \cap U_2) \to \tau_{U_1}(U_1 \cap U_2)$$

are $\mathbb{C}$-linear on the fibres, $\{z\} \times \mathbb{C}^n$. The maps $\{\tau_{U_i}\}$ for a suitable open cover $\{U_i\}$ of X, satisfying the compatibility conditions above, will be called trivialisations of E.
A holomorphic m-dimensional vector subbundle E' of E, $m \leq n$, is a submanifold of E such that $\pi(E') = X$ and for any $x \in X$, $\pi-1(x) \cap E$ is an m-dimensional vector subspace of $\{x\} \times \mathbb{C}^n$.
A (holomorphic, differentiable, continous) section of a holomorphic vector bundle $\phi : E \to X$, is a (holomorphic, differentiable, continous) map $\sigma : X \to E$ such that $\pi \circ \sigma = Id_X$.

An equivalent description for E can be given by simply specifying an open cover $\{U_i\}$ of X and for i, j such that $U_i \cap U_j \neq \emptyset$, an invertible $n \times n$ matrices g_{ij} whose entries are holomorphic functions on $U_i \cap U_j$ under the further condition that if $U_i \cap U_j \cap U_k \neq \emptyset$ then

$$g_{ij} g_{jk} g_{ki} = Id_{U_i \cap U_j \cap U_k}.$$

In the category of holomorphic vector bundles, all standard linear algebra operations, e.g. direct sums, products, exterior powers, symmetric powers, quotients (by holomorphic sub-bundle) are admissible.
Since we are particularly interested in forms on Kähler manifolds, especially those induced by metrics, we recall also the following construction.
Suppose we are given a line bundle $\mathcal{L}$ (i.e. a vector bundle whose fibres have dimension 1 over $\mathbb{C}$) on a compact Kähler manifold X. Let $\{U_i\}$ be an open cover of X such that $\mathcal{L}_{|U_i}$ admits a holomorphic trivialisation $\mathcal{L}_{|U_i} \cong U_i \times \mathbb{C}$. Such a trivialisation is equivalent to giving an everywhere non-zero holomorphic section σ_i of $\mathcal{L}$ on U_i (the one which can be identified with the constant section equal to 1 in the trivialisation). The transition

matrices g_{ij} corresponding to these trivialisations are given by invertible holomorphic functions on $U_i \cap U_j$. Obviously, we have

$$\sigma_i = g_{ij}\sigma_j$$

on $U_i \cap U_j$. Now, let h be a Hermitian metric on $\mathcal{L}$. For $x \in X$, $h(x)$ is clearly determined by its value on any non-zero element of $\mathcal{L}_x$, since $h_x(\lambda u) = |\lambda|^2 h_x(u)$. Set $h_i = h(\sigma_i)$. It is a strictly positive function on U_i

$$h_i = |g_{ij}|^2 h_j$$

on $u_i \cap U_j$. The 2-forms

$$\omega_i = \frac{1}{2\pi i}\partial\bar{\partial}\log h_i$$

on U_i thus coincide on $U_i \cap U_j$, since $\partial\bar{\partial}\log|g_{ij}|^2 = 0$, and provide a 2-form ω on X. The forms constructed in this way are clearly closed, since they are locally exact, and real of type $(1,1)$.

Definition 1.4.2. *The form ω just constructed will be called the Chern form relative to the metric h on $\mathcal{L}$.*

1.4.2 Projective bundles

The Fubini-Study metric on $\mathbb{P}^r$ generalises to projective bundles, and makes it possible to show that a projective bundle (coming from a vector bundle) over a Kähler manifold X is also a Kähler manifold.

Definition 1.4.3. *Let E be a holomorphic vector bundle of rank $r+1$ over a complex manifold X. The manifold $\mathbb{P}(E)$, which is the quotient of E minus the zero section by the natural fibrewise action of $\mathbb{C}^*$, is called the projective bundle associated to E.*

The complex structure on $\mathbb{P}(E)$ is obvious: $\mathbb{P}(E)$ admits a natural morphism π to X, which can be deduced from that of E by passing to the quotient. On open sets $\{U_i\}$ of a trivialisation of E, we have[1]

$$\pi^{-1}(U_i) \cong_i U_i \times \mathbb{P}^n,$$

and the identifications between

$$\pi^{-1}(U_i \cap U_j) \cong_i U_i \cap U_j \times \mathbb{P}^n$$

and

$$\pi^{-1}(U_i \cap U_j) \cong_j U_i \cap U_j \times \mathbb{P}^n$$

are given by the projective morphisms induced by the transition matrices of E, and are thus holomorphic.

[1]The subscript i means that the trivialisation of the projective bundle depends on the open set choosen.

There is a natural relative version of the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ defined on an n-dimensional projective space. Recall that $\mathcal{O}_{\mathbb{P}^n}(1)$ is given by the dual of the tautological bundle on $\mathbb{P}^n$.
Let now S be the line subbundle of π^*E over $\mathbb{P}(E)$ whose fibre at a point (x, v), $v \in \mathbb{P}(E_x)$ is the line l contracted on $\mathbb{P}(E_x)$ in v. We then define $\mathcal{O}_{\mathbb{P}(E)}(1)$ as the dual of S. On each fibre of $\mathbb{P}(E)$, naturally isomorphic to $\mathbb{P}^r$, the restriction of $\mathcal{O}_{\mathbb{P}(E)}(1)$ is naturally isomorphic to $\mathcal{O}_{\mathbb{P}^r}(1)$.
Let h be a Hermitian metric on the bundle E. Then h induces a Hermitian metric on π^*E and thus, by restriction, a metric on S and its dual $\mathcal{O}_{\mathbb{P}(E)}(1)$, $h_{\mathcal{O}_{\mathbb{P}(E)}(1)}$. Summing up to the Kähler form of X an appropriate multiple of the Chern form of $h_{\mathcal{O}_{\mathbb{P}(E)}(1)}$, we can prove the following

Theorem 1.4.4. *If X is compact Kähler and E is a holomorphic bundle over X, then the manifold $\mathbb{P}(E)$ is Kähler.*

Remark 1.4.5. It is a direct consequence of its construction that $\mathbb{P}(E)$ is compact: in fact, we can consider it as a quotient of the bundle of unit spheres of E for any Hermitian metric on E.

1.4.3 Blow-ups

Let X be a complex manifold, and $Y \subset X$ a complex submanifold of codimension k. Locally along Y, there exist holomorphic functions $f_1, \ldots, f_k$ with independent differentials, such that locally Y is given by the by the point satisfying the equations

$$f_1(z) = \cdots = f_k(z) = 0.$$

These equations are not unique, but we have the following relations:

Lemma 1.4.6. *If $g_1, \ldots, g_k$ form another system of local equations for Y, then locally in a neighbourhood of Y, there exists a matrix M_{ij} of holomorphic functions such that*

$$g_i = \sum_j M_{ji} f_j.$$

Moreover, the matrix M_{ij} is invertible along Y, and its restriction to Y is uniquely determined by the f_i, g_j.

Let U be an open set of X, on which there exist functions $f_1, \ldots, f_k$ with independent differentials such that

$$Y \cap U = \{z \in U \mid f_i(z) = 0,\ i = 1, \ldots, k\}.$$

Now set

$$\widetilde{U}_Y := \{(Z, z) \in \mathbb{P}^{k-1} \times U \mid Z_i f_j(z) = Z_j f_i(z),\ 1 \leq i, j \leq k\}. \tag{1.5}$$

Here, $Z = (Z_1, \ldots, Z_k)$ is a representative vector of the corresponding point of $\mathbb{P}^k$. $\widetilde{U}_Y$ is a smooth complex complex submanifold of $\mathbb{P}^{k-1} \times U$. We have a map $\tau_U = pr_2 : \widetilde{U}_Y \to U$ which is an isomorphism over $U \setminus (Y \cap U)$. Above $Y \cap U$, the fibre of pr_2 is equal to $\mathbb{P}^{k-1}$. It is now easy to show that given an open cover $\{U_i\}$ of Y, the blown up open sets $\{\widetilde{U}_{iY}\}$ glue together (clearly, where they have common intersections) to construct the blow-up of X along Y.
More precisely, it can be verified that given two open sets $U, V \subset X$, $U \cap V \cap Y \neq \emptyset$ in which Y is defined by equations $f_1^U, \ldots, f_k^U$ and $f_1^V, \ldots, f_k^V$ respectively, if $\tau_U : \widetilde{U}_Y \to U$ and $\tau_V : \widetilde{V}_Y \to V$ are the blow-ups of U and V along $U \cap Y$ and $V \cap Y$ respectively, there exists a natural isomorphism

$$\phi_{UV} : \tau_U^{-1}(U \cap V) \to \tau_V^{-1}(U \cap V)$$

such that $\tau_U = \tau_V \circ \phi_{UV}$. Indeed, it suffices to construct the isomorphism locally in the neighbourhood of $\tau_U^{-1}(Y \cap U)$, since such an isomorphism is certainly unique by continuity, and is already defined outside Y. Thus, up to restricting U, we can assume that we have a holomorphic invertible matrix M^{UV} which sends the equations f_i^U to the equations f_i^V (as in the previous lemma), i.e.

$$f_i^U = \sum_j M_{ji}^{UV} f_j^V .$$

Let now $P^{UV} = (M^{UV})^{-t}$. Then the biholomorphism

$$\psi_{UV} : \mathbb{P}^{k-1} \times (U \cap V) \to \mathbb{P}^{k-1} \times (U \cap V)$$

defined by $\psi_{UV}(Z, z) = (P^{UV}(z) \cdot Z, z)$ (where Z is considered as a column vector) clearly sends $\tau_U^{-1}(U \cap V)$ to $\tau_V^{-1}(U \cap V)$ and the inverse map is given by the inverse diffeomorphism.

Definition 1.4.7. *The manifold $\widetilde{X}_Y$ obtained by gluing together the manifolds $\{\widetilde{U}_{iY}\}$ above the common intersections $U_i \cap U_j$, is called the blow-up of X along Y. We will often call $\tau^{-1}(Y)$ the exceptional divisor of the blow-up (the reason will be clear in a moment), while we will sometimes refer to X as the base-space of the blow-up.*

We have a blow-up map $\tau : \widetilde{X}_Y \to X$, equal to τ_U over $\widetilde{U}_Y$. It is an isomorphism above $X \setminus Y$. We also have $\tau^{-1}(Y) = \mathbb{P}(N_{Y/X})$, since the matrices M_{ij} which give the transition morphisms for the projective bundle $\tau^{-1}(Y)$ are the transition matrices for the normal bundle $N_{X/Y} = T_{X|Y}/T_Y$. We easily see that $\tau^{-1}(Y) \subset \widetilde{X}_Y$ is a smooth hypersurface, i.e. a smooth complex submanifold of codimension 1. In fact, considering the local definition of the blow-up, if $(y, [Z_1 : \cdots : Z_k]) \in \widetilde{U}_Y, y \in Y$, then there exists Z_i such that $Z_i \neq 0$. It is easy to show that the function $f_i \circ \tau$ gives a local holomorphic equation for $\tau^{-1}(Y)$ in $\widetilde{U}_Y$ in the neighbourhood of $(y, [Z_1 : \cdots : Z_k])$. Since $\tau^{-1}(Y)$ is an hypersurface, it gives over $\widetilde{X}_Y$ an effective divisor. To that divisor, we can associate a well defined line bundle $\mathcal{L}$, trivial outside $\tau^{-1}(Y)$, whose transitions functions are given by the equations that cut out $\tau^{-1}(Y)$ over $\widetilde{X}_Y$ (this line bundle is usually indicated

by the symbol $\mathcal{O}_{\widetilde{X}_Y}(-\tau^{-1}(Y)))$. It is a well known fact that the restriction of $\mathcal{L}$ to $\tau^{-1}(Y)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(N_{Y/X})}(1)$. This fact is very useful, in order to understand the cohomology of a blow-up in terms of the class of the exceptional divisor and of the base space.
The first fundamental result abuot blow-ups of Kähler manifolds is the following:

Theorem 1.4.8. *If X is Kahler and $Y \subset X$ is a compact complex submanifold of X, the blown-up manifold $\widetilde{X}_Y$ is Kähler, and it is compact if X is.*

1.4.4 Hodge structure of a blow-up

Let X be a Kähler manifold, and let $Z \subset X$ be a submanifold. We know that the blow-up $\tau : \widetilde{X}_Z \to X$ of X along Z is still a Kähler manifold. Let $E := \tau^{-1}(Z) \cong \mathbb{P}(N_{Z/X})$ be the exceptional divisor. As we have seen E is a projective bundle of rank $r-1$, $r = codimZ$. In view of this fact, we have on E the line bundle $\mathcal{O}_E(1)$, defined above. Moreover, E is a smooth hypersurface of $\widetilde{X}_Z$. Let us denote by $j : E \hookrightarrow \widetilde{X}_Y$ the inclusion map. The Hodge structure on $H^k(\widetilde{X}_Z, \mathbb{Z})$ is described as follows.

Theorem 1.4.9. *Let $h = c_1(\mathcal{O}_E(1)) \in H^2(E, \mathbb{Z})$. Then we have a isomorphism of Hodge structure*

$$\tau^* + \sum_i \phi_i \quad : \quad H^k(X, \mathbb{Z}) \bigoplus_{i=0}^{r-2} H^{k-2i-2}(Z, \mathbb{Z}) \to H^k(\widetilde{X}_Z, \mathbb{Z}), \tag{1.6}$$

where ϕ_i is given by $j_* \circ h^i \circ \tau^*_{|E}$.

Here, h^i is the morphism of Hodge structure given by the cup-product by $h^i \in H^{2i}(E, \mathbb{Z})$. On the components H^{k-2i-2} of the left-hand term, we put the Hodge structure of Z shifted by $(i+1, i+1)$ in bidegree, so as to obtain a Hodge structure of weight k.
Sketch of the proof. By the results of the preceding section, the morphism (1.6) is a morphism of Hodge structure. It thus suffices to prove that it is an isomorphism of $\mathbb{Z}$-modules. Let $U \subset X$ be defined as

$$U := X \setminus Z.$$

Then U is also isomorphic to the open set $\widetilde{X}_Z \setminus E$ of $\widetilde{X}_Z$. As τ gives a morphism between the pair $(\widetilde{X}_Z, U)$ and the pair (X, U), we have a morphism τ^* between the long exact sequences of cohomology relative to these pairs (where we are considering cohomology groups with integral coefficients):

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H^{k-1}(U) & \longrightarrow & H^k(X,U) & \longrightarrow & H^k(X) & \longrightarrow & H^k(U) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \tau^*_U} & & \downarrow{\scriptstyle \tau^*_{X,U}} & & \downarrow{\scriptstyle \tau^*_X} & & \downarrow{\scriptstyle \tau^*_U} & & \\ \cdots & \longrightarrow & H^{k-1}(U) & \longrightarrow & H^k(\widetilde{X}_Z,U) & \longrightarrow & H^k(\widetilde{X}_Z) & \longrightarrow & H^k(U) & \longrightarrow & \cdots \end{array} \tag{1.7}$$

The first and last maps are of course the identity. Furthermore, by excision and by the Thorn isomorphism (see [BT82] or [Hat02]), we have

$$H^k(X, U) \cong H^{k-2r}(Z),\ H^k(\widetilde{X}_Z, U) \cong H^{k-2}(E).$$

Moreover the morphism $H^k(\widetilde{X}_Z, U) \cong H^{k-2}(E) \to H^k(\widetilde{X}_Z)$ can be identified with j_*, and the morphism $H^k(X, U) \cong H^{k-2r}(Z) \to H^k(X)$ can be identified with j_{Z*}, where j_Z is the inclusion of Z in X. Now, using Leray-Hirsch Theorem ([BT82]), we have that the cohomology $H^*(E, \mathbb{Z})$ of the projective bundle $E \to Z$ is a free module over the ring $H^*(Z, \mathbb{Z})$ with basis $1, h, \ldots, h^{r-1}$. We can give a more compact description as follows: there is a isomorphism of graded modules (where the grading is naturally induced by the degree in cohomology)

$$H^*(E, \mathbb{Z}) \to H^*(Z, \mathbb{Z})[h]/(h^r + \sum_{i=1}^{r} (-1)^i c_i(N_{Y/X}) h^{r-i}),$$

where we are treating the cohomology class h as a variable and $c_i(N_{Y/X}) \in H^{2i}(E, \mathbb{Z})$ is the i-th Chern class of the normal bundle.
We now conclude as follows: since τ has degree 1, then $\tau_X^* : H^k(X) \to H^k(\widetilde{X}_Z)$: is injective. The description of the cohomology of E implies that

$$\tau_{X,U}^* : H^k(X, U) \to H^k(\widetilde{X}_Z, U)$$

is injective: more precisely, we can consider $\tau_{X,U}^*$ as a morphism which we denote by

$$\alpha : H^{k-2r}(Z) \to H^{k-2}(E) = \bigoplus_{i=0}^{r-1} h^i \tau^* H^{k-2-2i}(Z).$$

It is not difficult to see that the $(r-1)$-th component α_{r-1} of α is equal to $h^{r-1}\tau^*$, and thus $\tau_{X,U}^*$ is indeed injective.
The commutativity of the diagram of long exact sequences of relative cohomology (1.4.9) then implies that the natural map

$$(\tau^*, j_*) : H^k(X) \oplus H^{k-2}(E) \to H^k(\widetilde{X}_Z)$$

is surjective, and the injectivity of $\tau_{X,U}^*$ in degree $k-1$ shows that the kernel of this map is

$$Im(j_{Z*}, -\alpha) : H^{k-2r}(Z) \to H^k(X) \oplus H^{k-2}(E).$$

From the fact that $\alpha_{r-1} = -h^{r-1}\tau^*$ it follws that (1.6) is an isomorphism.

∎

Remark 1.4.10. Let Y be an n-dimensional Kähler manifold, $E \subset Y$ a compact submanifold of codimension $k \geq 2$. Consider $\widetilde{Y}_E$, the blow-up of Y along E, and let $\widetilde{E}$ be exceptional divisor of $\widetilde{Y}_E$.

The blow-down map $\tau : \widetilde{Y}_E \to Y$ gives an injective map $\tau^* : H^*(Y, \mathbb{Z}) \to H^*(\widetilde{Y}_E, \mathbb{Z})$. We want to analyze the behaviour of the map

$$[\widetilde{E}] \cup \tau^* : H^*(Y, \mathbb{Z}) \to H^{*+2}(\widetilde{Y}_E, \mathbb{Z}).$$

Let $j_{\widetilde{E}} : \widetilde{E} \hookrightarrow \widetilde{Y}_E$, $j_E : E \hookrightarrow Y$ be the inclusion maps of the respective submanifolds and $\tau_{|\widetilde{E}} : \widetilde{E} \to E$ the restriction of the blow-down map to the exceptional divisor. The map $\tau_{|\widetilde{E}}$ gives $\widetilde{E}$ a structure of $\mathbb{P}^{n-k-1}$-bundle over E.
We claim that the map $[\widetilde{E}] \cup \tau^*$ can be written in the form

$$j_{\widetilde{E}*} \circ \tau^*_{|\widetilde{E}} \circ j^*_E. \tag{1.8}$$

Indeed, let Ψ be a De-Rham rapresentative of the fundamental class of $\widetilde{E}$ in the cohomology of $\widetilde{Y}_E$ (e.g. we could take a De-Rham rapresentative of the Thom class of the normal bundle of $\widetilde{E}$ in $\widetilde{Y}_E$); we have the following relation:

$$\int_{\widetilde{Y}_E} \Psi \wedge \alpha = \int_{\widetilde{E}} \alpha_{|\widetilde{E}} = \int_{\widetilde{E}} j^*_{\widetilde{E}}(\alpha),$$

where α is a closed form of degree $2n-2$ on $\widetilde{Y}_E$.
Hence we have

$$\begin{aligned} < [\widetilde{E}] \cup \tau^*([\beta]), [\sigma] >= \int_{\widetilde{Y}_E} \Psi \wedge \tau^*(\beta) \wedge \sigma = \int_{\widetilde{E}} j^*_{\widetilde{E}}(\tau^*(\beta)) \wedge j^*_{\widetilde{E}}(\sigma) = \\ =< j^*_{\widetilde{E}}(\tau^*([\beta])), j^*_{\widetilde{E}}([\sigma]) >_{|\widetilde{E}} =< j_{\widetilde{E}*}(j^*_{\widetilde{E}}(\tau^*([\beta]))), [\sigma] >, \end{aligned} \tag{1.9}$$

where $< \cdot, \cdot >$ represents the Poincaré duality on $\widetilde{Y}_E$, while $< \cdot, \cdot >_{\widetilde{E}}$ represents the Poincaré duality on $\widetilde{E}$ and $[\alpha] \in H^j(Y, \mathbb{Z})$, $[\sigma] \in H^{2n-2-j}(\widetilde{Y}_E, \mathbb{Z})$. From (1.9) we deduce that

$$[\widetilde{E}] \cup \tau^* = j_{\widetilde{E}*} \circ j^*_{\widetilde{E}} \circ \tau^*.$$

Now, since

$$j^*_{\widetilde{E}} \circ \tau^* = \tau^*_{|\widetilde{E}} \circ j^*_E,$$

by the commuting diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{j_{\widetilde{E}}} & \widetilde{Y}_E \\ \downarrow{\scriptstyle \tau_{|\widetilde{E}}} & & \downarrow{\scriptstyle \tau} \\ E & \xrightarrow{j_E} & Y, \end{array}$$

(1.8) is proved.
We are interested in finding out the kernel of (1.8). From the proof of (1.4.9), the composition $j_{\widetilde{E}*} \circ \tau^*_{|\widetilde{E}}$ is an injective map, hence

$$\ker j_{\widetilde{E}*} \circ \tau^*_{|\widetilde{E}} \circ j^*_E = \ker\ j^*_E. \tag{1.10}$$

This formula will often be used in the next chapters.

1.5 The Kodaira problem

1.5.1 Analytic spaces and deformations

We start with the definition of analytic set.

Definition 1.5.1. *Let M be a complex analytic manifold. A subset $A \subset M$ is said to be an analytic subset of M if A is closed and if for every point $x_0 \in A$ there exist a neighborhood U of x_0 and holomorphic functions $g_1, \dots, g_n \in \mathcal{O}(U)$ such that $A \cap U = \{z \in U \mid g_i(z) = 0,\ i = 1, \dots, n\}$. Then g_i are said to be (local) equations of A in U.*

In particular we will focus on analytic subsets of open sets $\Omega \subset \mathbb{C}^n$. We also define the structure sheaf of an analytic set.
Given an analytic set $A \subset \Omega \subset \mathbb{C}^n$, we have that the local equations for A, in a neighbourhood U such that $U \cap A \neq \emptyset$, give an ideal $\mathcal{I}_{A \cap U_x}$ in the ring $\mathcal{O}_U$ of holomorphic functions in U. Hence, we take the radical of such an ideal $\sqrt{\mathcal{I}_{A\cap U}} \subset \mathcal{O}_U$. We define the structure sheaf of A in the open set $A \cap U$ as the quotient

$$\mathcal{O}_{A\cap U} := \mathcal{O}_U / \sqrt{\mathcal{I}_{A\cap U}}.$$

We can repeat this operation on an open cover $\{U_j\}$ of A. It is not difficult to see that the sheaves of rings $\{\mathcal{O}_{A\cap U_i}\}$ are indipendent by the choice of local equations for A in the U_i's and that if $U_i \cap U_j \neq \emptyset$ then the restrictions of $\mathcal{O}_{A\cap U_i}$ and $\mathcal{O}_{A\cap U_j}$ on $U_i \cap U_j$ are equal. Hence we can glue this sheaves and we obtain a well defined sheaf of rings on A. Moreover, this sheaf is indipedent by the choiche of the open cover $\{U_j\}$.

Definition 1.5.2. *The sheaf of rings just defined will be called the structural sheaf of A, $\mathcal{O}_A$*

It is clear that if x is a point of A and in a neighbourhood U_x of x, A is given by holomorphic equations $g_i = 0$, $i = 1, \dots, n$, then, the stalk of $\mathcal{O}_A$ at x is nothing but the localization at x of the ring $\mathcal{O}_{U_x}/\sqrt{(g_i)}$ Now that we have the complete definition of an analityc set, we want to define also a good class of morphism.

Definition 1.5.3. *Let $A \subset \Omega_A \subset \mathbb{C}^a$, $B \subset \Omega_B \subset \mathbb{C}^b$ be analytic sets. A morphism from A to B is by definition a continuous map $F : A \to B$ such that for every $x \in A$ there is a neighborhood U of x and a holomorphic map $\tilde{F} : U \to \mathbb{C}^b$ such that $\tilde{F}_{|U\cap A} = f_{|U\cap A}$. Equivalently, such a morphism can be defined as a continuous map $F : A \to B$ such that for all $x \in A$ and $g \in \mathcal{O}_{B,F(x)}$, we have $g \circ F \in \mathcal{O}_{A,x}$. The induced ring morphism*

$$F_x^* : \mathcal{O}_{B,F(x)} \to \mathcal{O}_{A,x}$$

is called the comorphism of F at point x.

The next step is to define complex analytic spaces. These will be defined, as it often happens in geometry, as spaces that locally look like special sets, in this case analytic sets in $\mathbb{C}^n$.

Definition 1.5.4. *A complex analytic space X is a locally compact separable (i.e. possessing a countable basis of the topology) Hausdorff space, together with a sheaf $\mathcal{O}_X$ of continuous functions on X, such that there exists an open covering $\{U_i\}$ of X and for each U_i a homeomorphism $F_i : U_i \to A_i$ onto an analytic set $A_i \subset \mathbb{C}^{n_i}$ such that the comorphism $F_i^* : \mathcal{O}_{A_i} \to \mathcal{O}_{X|U_i}$, $\mathcal{O}_{A_i} \ni g \mapsto g \circ F$, is an isomorphism of sheaves of rings. $\mathcal{O}_X$ is called the structure sheaf of X. A complex analytic space X is said to be proper if X is compact.*

An analytic subset A of a complex analytic space X will be a closed set such that on an open cover $\{U_i\}$, with the same properties as in previous definition, $A \cap U_i$ is an analytic subset. A complex space X is said to be irreducible if it is not a finite union of analytic subsets X_i, $X_i \subsetneq X$.

Remark 1.5.5. Analytic spaces share many of the properties of analytic sets. In particular they have a decomposition in irreducible analytic subsets, i.e. for any complex analytic space X, there exist a finite collection of analytic subsets $\{X_i\}$ such that

$$X = \bigcup_i X_i$$

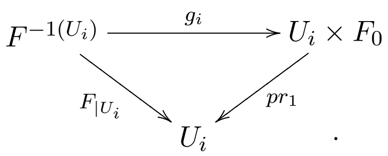
and such decomposition is unique up to a permutation of the X_i's. We can also define the dimension of an irreducible complex space X, analogously to the concept of dimension in classical algebraic geometry. For a more complete treatment of analytic spaces see [GH78] and [Dem].

Morphisms of compex analytic spaces are defined as morphism that locally on an open cover are morphism of analytic sets.
We are particularly interested in some properties of morphism of analytic sets.

Definition 1.5.6. *Let $F : X \to Y$ a morphism of analytic sets. F is said to be proper if F is proper as a continuos map between X and Y, i.e. if the counterimage of compact subsets of Y is compact.*
A morphism $G : X \to Y$ of irreducible analytic sets is said to be smooth if G is flat, i.e. $\forall x \in X$, $\mathcal{O}_{X,x}$ is flat as $\mathcal{O}_{Y,f(x)}$-algebra and the fibre of G are smooth connected manifolds.

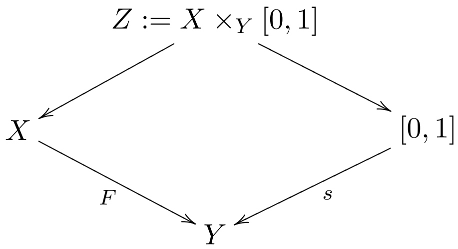
Thus, by definition, if $F : X \to Y$ is a smooth morphism of analytic spaces, then the fibre of F over any point $y \in Y$, $F^{-1}(y)$ is a smooth complex manifold. If, moreover, F is also proper, then $F^{-1}(y)$, will be a compact complex manifold, for any $y \in Y$.
Next we have the following theorem due to Ehresmann

Theorem 1.5.7. *Let $F : X \to Y$ be a proper submersion (i.e. $\forall x \in X$ the differential $dF_x : T_xX \to T_{f(x)}Y$ is a surjective morphism of vector spaces) of C^∞ manifolds. Then F is a fibration, i.e. there is an open cover $\{U_i\}$ of Y and diffeomorphism $g_i : F^{-1}(U_i) \to U_i \times F_0$, where F_0 is the manifold $F_0 := f^{-1}(0)$, $0 \in Y$ such that the diagram below* *commutes*



Here pr_1 is the projection on the first factor of the cartesian product.

Hence if $F : X \to Y$ is a proper smooth morphism of complex spaces, fixing a point $0 \in Y$ we can consider the manifold $X_0 := F^{-1}(0)$. We want to understand how the complex and differentiable structures of X_0 change, when the point t vaires in Y.
By considering a curve $s : [0,1] \to Y$ connecting a point $t \in Y$ to 0, we can consider the fibre product



and it is easy to show that Z can be given the structure of smooth manifold, inducing the right differentiable structure on fibres of F. Thus, by means of Ehresmann's theorem we see that all the fibres on F are diffeomorphic. The same is not true in the category of complex manifolds and that is an important point. In the following paragraphs and chapters, we will use this point of view to discuss the Kodaira problem.

Definition 1.5.8. *Given a compact Kähler manifold X, a deformation of X is the data of $(\mathcal{X}, \pi, \mathcal{B}, 0)$ where $\mathcal{X}, \mathcal{B}$ are analytic spaces, π is a proper smooth morphism between $\pi : \mathcal{X} \to \mathcal{B}$, $0 \in \mathcal{B}$ and we have a fixed biholomorphims between X and $X_0 = \pi^{-1}(0)$.*

Here we consider any deformation parameterized by a connected analytic space $\mathcal{B}$, that is any smooth proper map of analytic spaces $\pi : \mathcal{X} \to \mathcal{B}$, with $\pi^{-1}(0) = X$ for some $0 \in \mathcal{B}$, with $\mathcal{B}$ connected. Then any fiber X_t, $t \in \mathcal{B}$, will be said to be a deformation of X_0. We shall also say that X_t is deformation equivalente to X_0.

1.5.2 The Kodaira criterion

The Kodaira criterion (see [Kd54]) characterizes projective complex manifolds inside the class of compact Kähler manifolds.

Theorem 1.5.9. *A compact complex manifold X is projective if and only if X admits a Kähler class which is rational, i.e. that belongs to*

$$H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R}).$$

The "only if" comes from the fact that if X is projective, one gets a Kähler form on X by restricting the Fubini-Study Kähler form on some projective space $\mathbb{P}^N$ in which X is imbedded as a complex submanifold. But the Fubini-Study Kähler form has integral cohomology class, as its class is the first Chern class of the holomorphic line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ on $\mathbb{P}^N$. Conversely, if the class β of a Kähler form Ω is rational, some multiple $\alpha = m\beta$ is integral, and as α is represented by a closed form of type (1, 1), its image in $H^2(X, \mathcal{O}_X)$ vanishes. Thus, via the long exact sequence induced by the exponential exact sequence:

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathcal{O}_X),$$

one concludes that $\alpha = c_1(L)$ for some holomorphic line bundle L.
The conclusion then follows from the following two facts:

- L can be endowed with a Hermitian metric whose Chern form is equal to $m\Omega$, a non-trivial fact which involves the $\partial\bar{\partial}$-lemma, and uses the fact that X is Kähler.
- Kodaira's vanishing theorem for line bundles endowed with metrics of positive associated Chern forms, applied to the blow-up of X along points, which finally allow to conclude that L is ample.

Definition 1.5.10. *A polarization on a projective manifold X is the data of a rational Kähler cohomology class.*

As explained in the previous section, a polarization on X induces an operator L of cup-product with the given Kähler class and a Lefschetz decomposition on each cohomology group $H^k(X, \mathbb{Q})$ and a polarization on each component $L^r H^{k-2r}(X, \mathbb{Q})_{prim}$ of the Lefschetz decomposition, which is essential for most statements concerning the period map. The simplest application of Kodaira characterization of projective complex manifolds is the following

Theorem 1.5.11. *Let X be a compact projective Kähler manifold such that $H^2(X, \mathcal{O}_X) = 0$ then X is projective.*

This is a simple application of the fact that the set of Kähler classes in $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$ is a cone, which, if $H^2(X, \mathcal{O}_X) = 0$, is open.

1.5.3 Kodaira's theorem for surfaces

Kodaira's embedding theorem (1.5.9) can also be used to show that certain compact Kähler manifolds X become projective after a small deformation of their complex structure. The point is that Kähler classes belong to

$$H^{1,1}(X)_{\mathbb{R}} := H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{R}),$$

the set of degree 2 cohomology classes which can be represented by a real closed (1, 1)-form. They even form an open cone, the Kähler cone, in this real vector subspace

of $H^2(X, \mathbb{R})$. This subspace deforms differentiably with the complex structure, and by Kodaira's criterion we are reduced to see whether one can arrange that after a small deformation of the complex structure on X, the deformed Kähler cone contains a rational cohomology class.
Let us state the beautiful theorem of Kodaira which was at the origin of this circle of problems.

Theorem 1.5.12 ([Kd63])**.** *Let S be a compact Kähler surface. Then there is an arbitrarily small deformation of S which is projective.*

By an arbitrarily small deformation, we mean here that there is a deformation of S, $(X, \pi, \Delta, 0)$, where Δ is the open unit disk in $\mathbb{C}$ and 0 is the origin and there are points $t_n \to 0, n \in \mathbb{N}$ in Δ for which $\pi^{-1}(t_n)$ is a projective manifold. Kodaira proved this theorem using his classification of surfaces. Recently, Buchdahl (see [Bu06], [Bu08]) gave a proof of Kodaira Theorem which does not use the classification. His proof uses techniques of analytical geometry and shows for example that a rigid compact Kähler surface is projective.

1.5.4 Various aspects of the Kodaira problem

Kodaira's theorem (1.5.12) leads immediately to ask a number of questions in higher dimensions:

Question 1 (The Kodaira problem)**.** *Does any compact Kähler manifold admit an arbitrarily small deformation which is projective?*

In order to disprove this, it suffices to find rigid Kähler manifolds which are not projective. However, the paper [DEP05] shows that it is not so easy. The authors show that a complex torus T, carrying three holomorphic line bundles L_1, L_2, L_3 such that the deformations of T preserving the L_i are trivial, is projective. The relation with the previous problem is the fact that from (T, L_1, L_2, L_3), one can construct a compact Kähler manifolds whose deformations identify to the deformations of the 4-uple (T, L_1, L_2, L_3). Hence it is not so easy to find concrete examples of the rigidity phenomena above, that would contradict the Kodaira problem.
Thus, one can try to answer a weaker question concerning global deformations.

Question 2 (The global Kodaira problem)**.** *Does any compact Kähler manifold X admit a deformation which is projective?*

Here we consider any deformation parameterized by a connected analytic space $\mathcal{B}$, that is any smooth proper map of analytic spaces $\pi : \mathcal{X} \to \mathcal{B}$, with $\pi^{-1}(0) = X$ for some $0 \in \mathcal{B}$.
As explained in Section 1.5.1, for any $t \in \mathcal{B}$, X_t will be diffeomorphic to X. In that case, even the existence of rigid Kähler manifolds which are not projective would not suffice to provide a negative answer, as there exist varieties which are locally rigid but not globally

(consider for example the case of $\mathbb{P}^1 \times \mathbb{P}^1$which deforms to a different Hirzebruch surface). In particular, X and Y should be homeomorphic, hence have the same homotopy type, hence also the same cohomology ring. Thus Question 2 can be weakened as follows :

Question 3 (The topological Kodaira problem)**.** *Is any compact Kähler manifold X diffeomorphic or homeomorphic to a projective complex manifold? Does any compact Kähler manifold X have the homotopy type of a projective complex manifold?*

A part of the results outlined in this thesis will provide a negative answer to all these questions, answering negatively the last one.
Nevertheless, the examples built here have the property that they are bimeromorphically equivalent to complex tori or Kummer manifolds, which have small projective deformations (see [V02, II, Ch. 5, page 153]). Thus, a natural generalization of Question 3 is the following:

Question 4 (The birational Kodaira problem)**.** *Is any compact Kähler manifold X bimeromorphic to a smooth compact complex manifold which deforms to a projective complex manifold?*

We shall also exhibit examples of manifold contradicting also this question. However, the compact Kähler manifolds constructed there have negative Kodaira dimension, as they are bimeromorphic to $\mathbb{P}^1 \times \mathbb{P}^1$-bundles on a product of Kummer manifolds. Thus the following remains open:

Conjecture (Campana)**.** *Is any compact Kähler manifold X of nonnegative Kodaira dimension bimeromorphic to a smooth compact complex manifold which deforms to a projective complex manifold?*

Chapter 2

The Kodaira problem

The the so-called Kodaira problem asks:

Question. *Let X be a compact Kähler manifold. Is it always possible to find a deformation of X which is a complex projective manifold?*

It is a natural higher dimensional analogue of the 2-dimensional case which was solved affermatively by Kodaira in 1960 as stated in the Introduction. We shall always refer to the above question as the Kodaira problem.
Voisin showed in [V03] that it is possible to construct Kähler compact manifold X for which the answer to the Kodaira problem is negative.
In this chapter we examine in full detail these manifolds: more precisely, we show that

Theorem. *Given $n \geq 4$, it is always possible to find n-dimensional compact Kähler manifolds which have never the homotopy type of complex projective manifolds.*

In Section 2.1, we illustrate the basic construction of such counterexamples. These are particular blow-ups of tori along smooth submanifolds. We also prove the above theorem. In Section 2.2, we show an important lemma due to Deligne, which allows us to modify the examples given to obtain new examples, even simply connected ones.

2.1 Construction of a counterexample

Let $\Gamma \cong \mathbb{Z}^{2n}$ be a lattice and let $\phi : \Gamma \to \Gamma$ be an endomorphism. Let $\Gamma_{\mathbb{C}} := \Gamma \otimes \mathbb{C}$ be the complex vector space obtained by extending scalar multiplication and let $\phi_{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of ϕ. Throughout the chapter, we will assume that ϕ satisfies the following property:

($*$) *The eigenvalues of ϕ all have multiplicity 1 and none of them is real.*

Since the eigenvalues are all distinct, $\phi_{\mathbb{C}}$ can be diagonalized. We shall choose among the $2n$ eigenvalues of ϕ a subset of n of them, namely $\{\lambda_1, \ldots, \lambda_n\}$, such that $\lambda_i \neq \overline{\lambda_j}$, for $i, j \in \{1, \ldots n\}$.

The direct sum of the eigenspaces associated to the λ_i is a stable subspace for the action of $\phi_{\mathbb{C}}$: it will be denoted by Γ'. As an immediate consequence, we have

$$\Gamma_{\mathbb{C}} = \Gamma' \oplus \overline{\Gamma'}$$

and we can define a complex torus

$$T = \Gamma_{\mathbb{C}}/(\Gamma' \oplus \Gamma). \tag{2.1}$$

As $\phi_{\mathbb{C}}$ leaves both Γ and Γ' invariant, it induces and endomorphism of T, which we will denote ϕ_T.
First of all, we study the endomorphism ϕ_{T*} for the first homology group of T.
Recall, that we can identify canonically $H_1(T,\mathbb{Z})$ with Γ, through its projection on Γ'. It is immediate that the action of ϕ_{T*} on $H_1(T,\mathbb{Z})$ is the same as that of ϕ on Γ.

Remark 2.1.1. Conversely, given an n-dimensional complex torus T and an endomorphism ϕ_T, whose induced morphism $\phi_{T*} : H_1(T,\mathbb{Z}) \to H_1(T,\mathbb{Z})$ is the same as the one just described, we can identify T with one of the tori constructed above.
In fact, $T \cong \mathbb{C}^n/\Gamma$, where Γ is a lattice of dimension $2n$; on $\mathbb{C}^n$ we have the antilinear map j, induced by componentwise conjugation. Consider the torus $\widetilde{T} = \mathbb{C}^n/\Gamma \oplus \mathbb{C}^n/j(\Gamma)$: given a basis for Γ, namely $\{\alpha_1,\ldots,\alpha_{2n}\}$, $\{j(\alpha_1),\ldots,j(\alpha_{2n})\}$ is a basis for $j(\Gamma)$ and $\{\alpha_1 + j(\alpha_1),\ldots,\alpha_{2n} + j(\alpha_{2n})\}$ is a complex basis for $\mathbb{C}^n \oplus \mathbb{C}^n$; indeed, if there were $c_i \in \mathbb{C}$, $i = 1,\ldots,2n$ such that $\sum_i c_i[\alpha_i + j(\alpha_i)] = 0$ then

$$\sum_i c_i\alpha_i = \sum_i c_i j(\alpha_i) = \sum_i j(\overline{c_i}\alpha_i),$$

so that

$$\sum_i c_i\alpha_i = 0 = \sum_i j(\overline{c_i}\alpha_i)$$

and

$$\sum_i (c_i + \overline{c_i})\alpha_i = 0,$$

but since $c_i + \overline{c_i} \in \mathbb{R}, \forall i$ and the $\{\alpha_i\}$ are linearly indipendent over $\mathbb{R}$ that would imply that the real part of the c_i's is equal to $0, \forall i$. Since the same is true also for $c_i - \overline{c_i}$, the immaginary part of the c_i's is zero, hence $c_i = 0, \forall i$.
Projecting on $\mathbb{C}^n/\Gamma$, we can identify the lattice generated by the vectors $\{\alpha_1+j(\alpha_1),\ldots,\alpha_{2n}+j(\alpha_{2n})\}$ with a basis of $H_1(T,\mathbb{Z})$. We obtain an endomorphism acting on the lattice exactly as ϕ already described and we can reproduce the arguments above.

2.1.1 A first criterion of non-projectivity

The characteristic polynomial of ϕ is monic with integral coefficients and it is subject to the condition $(*)$. Moreover we shall assume that its Galois group is the greatest possilbe, namely S_{2n}[1], where $2n$ is the degree of the polynomial. A direct consequence

[1] We shall denote by S_{2n} the group of permutations of $2n$ objects.

of property $(*)$ and this assumption is the fact that T is not an abelian variety, as we will show below.

Remark 2.1.2. We want to show that it is always possible to find a polynomial satisfying these two assumptions (for any even degree).
Let us fix $n \in \mathbb{N}$. Choose now three different prime numbers $p_1, p_2, p_3 \in \mathbb{N}$. Let $f_1, f_2, f_3 \in \mathbb{Z}[X]$ be polynomials of degree $2n$ such that

- f_1 is irreducible modulo p_1;
- f_2 is the product of a linear factor and an irreducible polynomial of degree $2n-1$ modulo p_2;
- f_3 is the product of an irreducible quadratic polynomial and of $2n-2$ linear factors modulo p_3.

Let $g(X) \in \mathbb{Z}[X]$ be given by the equality

$$g(X) = p_2p_3f_1 + p_1p_3f_2 + p_1p_2f_3.$$

It can be easily proved that $Gal_{\mathbb{Q}}(g(X)) = S_{2n}$.
Now take a polynomial $h(X) \in \mathbb{Q}[X]$ with no real roots (e.g. $(x^2+1)^n$). Replacing the coefficients of h by close rationals will not create any real roots. So replace the X^k coefficient of h by a sufficient close rational a_k/b_k where a_k and b_k are congruent modulo $p_1p_2p_3$ to the X^k coefficient of g and to 1, respectively. Then the new polynomial has rational coefficients, no real roots and Galois group S_{2n}. Clearing denominators, we obtain a intergal polynomial $\tilde{h}(X) \in \mathbb{Z}[X]$. Let c_{2n} be its leading coefficient. Then it is not hard to show that

$$s(X) = c_{2n}^{2n-1}h(\frac{X}{c_{2n}})$$

is an intergal monic polynomial satisfying condition $(*)$.

Lemma 2.1.3. *If $n \geq 2$ and the characteristic polynomial of ϕ is Galois, i.e. the Galois group of its splitting fiels acts as the symmetric group on its roots, the torus T is not an abelian variety.*

Proof. Consider the subgroup of $H^2(T, \mathbb{Q})$ generated over $\mathbb{Q}$ by Chern classes of holomorphic line bundles on T, $NS_{\mathbb{Q}}(T)$. By a classical result in Hodge theory (see [V02, I, Ch. 11, Prop. 11.27]), $NS_{\mathbb{Q}}$ is contained in the subspace $H^{1,1}(T) \subset H^2(T, \mathbb{C})$ and it is stable under the action of ϕ_T^* on $H^2(T, \mathbb{Q})$: in fact, given a line bundle L on T and its Chern class $c_1(L) \in H^2(T, \mathbb{Q})$, the naturality of Chern classes gives

$$c_1(\phi_T^*(L)) = \phi_T^*(c_1(L)).$$

In the case of the torus, we have the following identifications (direct consequences of the universal coefficient theorem):

$$H^m(T, \mathbb{C}) = (H_m(T, \mathbb{C}))^* \tag{2.2}$$

$$\phi_T^* = (\phi_{T*})^* = \phi^t \tag{2.3}$$

so that the eigenvalues of ϕ_T^* on $H^{1,0}(T) \cong (\overline{\Gamma}')^*$ are the $\overline{\lambda_i}$ and the eigenvalues of ϕ_T^* on $H^{0,1}(T) \cong (\Gamma')^*$ are the λ_i.
We deduce that $H^2(T,\mathbb{C}) \cong \bigwedge^2 \Gamma^* \otimes \mathbb{C}$ completely decomposes into the direct sum of the eigenspaces relative to the action of ϕ_T^*, because this is true for its Hodge decomposition (ϕ_T^* being a morphism of Hodge structure)

$$\begin{aligned} H^2(T,\mathbb{C}) &= H^{2,0}(T) \oplus H^{1,1}(T) \oplus H^{0,2}(T) \cong \\ &\cong \bigwedge^2 (\overline{\Gamma}')^* \oplus ((\Gamma')^* \otimes (\overline{\Gamma}')^*) \oplus \bigwedge^2 (\Gamma')^*. \end{aligned}$$

and the eigenvalues of ϕ_T^* acting on $H^{1,1}(T) \cong \Gamma'^* \otimes \overline{\Gamma'}^*$ are the $\lambda_i\overline{\lambda_j}$'s, forall $i,j \in \{1,\dots,n\}$.
As $NS_{\mathbb{Q}}(T) \otimes \mathbb{C} \subset H^{1,1}(T)$ is stable under ϕ_T^*, we find couples $(i,j) \in \{1,\dots,n\} \times \{1,\dots,n\}$ such that $NS_{\mathbb{Q}}(T) \otimes \mathbb{C}$ is the direct sum of eigenspaces relative to eigenvalues of the form $\lambda_i\overline{\lambda_j}$.
For $NS_{\mathbb{Q}}(T) \otimes \mathbb{C}$ is defined over $\mathbb{Q}$, the action of the Galois group of the splitting field of the characteristic polynomial of ϕ leaves stable the eigenvalues relative to the couples (i,j). By $(*)$, the Galois group acts transitively on the roots of f, $\{\lambda_1,\dots,\lambda_n,\overline{\lambda_1},\dots,\overline{\lambda_n}\}$.
But $\dim_{\mathbb{C}} T \geq 2$ and we can find a permutation $\sigma \in S_{2n} = Gal(\mathbb{Q}[\lambda_1,\dots,\lambda_n,\overline{\lambda_1},\dots,\overline{\lambda_n}]/\mathbb{Q})$ such that $\sigma(\lambda_i,\overline{\lambda_j}) = (\lambda_k,\lambda_l)$, $k,l \in \{1,\dots,n\}$. This is not possible, because eigenvalues of the form $\lambda_k\lambda_l$ are relative to eigenvectors in $H^{0,2}(T)$, hence $NS_{\mathbb{Q}}(T) = \emptyset$.
As a consequence, T cannot be an abelian varieties, otherwise $NS_{\mathbb{Q}}(T) \otimes \mathbb{C}$ would be non-zero, containing the pullback of the Hodge class of the Fubini-Study metric of the projective space in which it would be immersed. □

Remark 2.1.4. Exactly the same reasoning shows us a deeper fact: there are no closed analytic subvarieties in T.
In fact, a closed analytic subvariety, T_1, of codimension r in T, would give rise to a non-zero cohomology class $[T_1] \in H^{r,r}(T,\mathbb{C}) \cap H^{2r}(T,\mathbb{Q})$. Considering the rational subspace generated by classes associated to codimension r analytic subsets (which is stable for the action of ϕ_T^*), and again noting that the action of S_{2n} does not fix any set of eigenvalues of ϕ_T^*, we see that such classes can not exist. Moreover, the same proof wuold work if instead we took the dual of ϕ (since the dual of a diagonal operator is the operator itself).
It is a consequence of this fact that T is also a simple torus.

Consider the complex manifold $T \times T$. We are interested in considering the following complex subtori:

- the diagonal $T_{\text{diag}} = \{(x,y) \in T \times T | x = y\}$,
- the graph of ϕ_T, $T_{\text{graph}} = \{(x,\phi_T(x)), x \in T\}$,
- $T \times \{0\} = \{(x,0), x \in T\}$,
- $\{0\} \times T = \{(0,x), x \in T\}$.

Proposition 2.1.5. *Given any two of the preceding submanifolds, they meet transversally at finitely many points.*

Proof. Clearly, $(T \times \{0\}) \cap (\{0\} \times T) = (\{0\} \times T) \cap (T_{\text{graph}}) = (T \times \{0\}) \cap (T_{\text{diag}}) = (\{0\} \times T) \cap (T_{\text{diag}}) = \{0\}$.
There are two more cases to analyze:

- $T \times \{0\} \cap T_{\text{graph}} = ker\phi$
Since the lifting of ϕ to $\mathbb{C}^n$ is invertible, we conclude that ϕ is an isogeny. Moreover, by the group law and the fact that the translation by a fixed point is a biholomorphism (since it is induced by the translation on the universal covering of the torus) we can reduce ourselves to the case of the intersection of the subtori at the point 0. Then, looking at the situation in $\mathbb{C}^n \oplus \mathbb{C}^n$ where n is the complex dimension of the torus T (here we are simply considering again the lifting $\mathbb{C}^n \oplus \mathbb{C}^n \to T \times T$), $T \times \{0\}$ is lifted to the complex n-subspace $\mathbb{C}^n \times \{0\}$ and T_{graph} is lifted to the n-subspace Graph $= \{(x, \phi(x)), x \in \mathbb{C}^n\}$. In order to prove the assertion, we have to verify that these two subspaces meet transversally at the origin, which is equivalent to verify that their sum (as complex vector subspaces) is $\mathbb{C}^n \oplus \mathbb{C}^n$. This is exactly our case: given a basis of $\mathbb{C}^n \times \{0\}$, $\{e_1, \dots, e_n\}$, then $\{(e_i, \phi(e_i)),\ i = 1, \dots, n\}$ is a basis for Graph, hence $\mathbb{C}^n \times \{0\}$ + Graph includes the subspace $\{0\} \times Im(\phi) = \{0\} \times \mathbb{C}^n$ and the claim is proved.

- $T_{\text{diag}} \cap T_{\text{graph}} = ker(\phi - Id_T)$
Exactly as above, we have that the cardinality of $T_{\text{diag}} \cap T_{\text{graph}}$ is finite (since also $\phi - Id_T$ is an isogeny on T, being an invertible operator on the lifting of the torus, since ϕ has complex non-real eigenvalues) and we can reduce ourselves to the case of the intersection of the subtori in 0. In $\mathbb{C}^n \oplus \mathbb{C}^n$, T_{diag} is lifted to the n-dimensional subspace Diag given by the diagonal immersion

$$\mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n,\ t \mapsto (t, t)$$

Thus at the origin we must see that Diag and Graph meet transversally. Indeed, we show that Diag + Graph $= \mathbb{C}^n \oplus \mathbb{C}^n$. Now, $\{(e_i, e_i),\ i = 1, \dots n\}$ is a complex basis for Diag, hence Diag + Graph includes the subspace $\{0\} \times Im(\phi - Id) = \{0\} \times \mathbb{C}^n$ and consequently the subspace $\mathbb{C}^n \times \{0\}$. From this we deduce that also $\{0\} \times \mathbb{C}^n$ is in Diag + Graph and this ends the proof.

□

For each of these subtori, we consider the inclusion maps

$$\begin{aligned} j_{T\times\{0\}} &: T \times \{0\} \hookrightarrow T \times T, \\ j_{\{0\}\times T} &: \{0\} \times T \hookrightarrow T \times T, \\ j_{T_{\text{diag}}} &: T_{\text{diag}} \hookrightarrow T \times T, \\ j_{T_{\text{graph}}} &: T_{\text{graph}} \hookrightarrow T \times T. \end{aligned}$$

Clearly, each of these maps gives homomorphism in cohomology

$$\begin{aligned} j^*_{T\times\{0\}} &: H^*(T\times T,\mathbb{Z}) \to H^*(T\times\{0\},\mathbb{Z}),\\ j^*_{\{0\}\times T} &: H^*(T\times T,\mathbb{Z}) \to H^*(\{0\}\times T,\mathbb{Z}),\\ j^*_{T_{\text{diag}}} &: H^*(T\times T,\mathbb{Z}) \to H^*(T_{\text{diag}},\mathbb{Z}),\\ j^*_{T_{\text{graph}}} &: H^*(T\times T,\mathbb{Z}) \to H^*(T_{\text{graph}},\mathbb{Z}). \end{aligned} \tag{2.4}$$

Particularly, we will focus on the first cohomology groups.
Recall that we have the Künneth decomposition

$$H^1(T\times T,\mathbb{Z}) \cong H^1(T,\mathbb{Z}) \oplus H^1(T,\mathbb{Z}) \tag{2.5}$$

where the isomorphism is given by the pull-back maps of the projections

$$T \xleftarrow{\ \pi_1\ } T\times T \xrightarrow{\ \pi_2\ } T$$

Hence by (2.5) there are maps

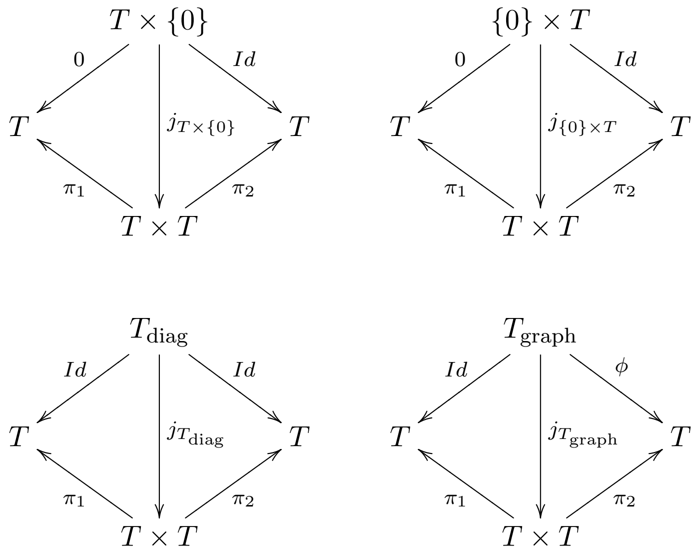
$$H^1(T,\mathbb{Z}) \xleftarrow{\ pr_1\ } H^1(T\times T,\mathbb{Z}) \xrightarrow{\ pr_2\ } H^1(T,\mathbb{Z})$$

We want to study the kernel of maps in (2.4), in terms of the cohomology maps $pr_1,\ pr_2$. The result is the following

Lemma 2.1.6.

$$\begin{aligned} Ker\big\{j^*_{T\times\{0\}} : H^*(T\times T,\mathbb{Z}) \to H^*(T\times\{0\},\mathbb{Z})\big\} &= Ker\ pr_1,\\ Ker\big\{j^*_{\{0\}\times T} : H^*(T\times T,\mathbb{Z}) \to H^*(\{0\}\times T,\mathbb{Z})\big\} &= Ker\ pr_2,\\ Ker\big\{j^*_{T_{diag}} : H^*(T\times T,\mathbb{Z}) \to H^*(T_{diag},\mathbb{Z})\big\} &= Ker\ pr_1 + pr_2,\\ Ker\big\{j^*_{T_{graph}} : H^*(T\times T,\mathbb{Z}) \to H^*(T_{graph},\mathbb{Z})\big\} &= Kerpr_1 + \phi^*_T \circ pr_2. \end{aligned}$$

Proof. All the above assertions are consenquences of the universal property of product, that induces the following commutative diagrams:



□

2.1.2 The fundamental Theorem

Blowing up the points at which the four submanifolds meet, we obtain a new Kähler complex manifold (see Theorem (1.4.8)), $\widetilde{T \times T}$, in which the proper tranforms of the four subtori are disjoint. Blowing up their union, we finally get a compact manifold, X, once again Kähler.
The following fundamental theorem on X will be the main result of this chapter.

Theorem 2.1.7. *Assume $n \geq 2$ and that the characteristic polynomial of ϕ is Galois. Let X' be a compact Kähler manifold and assume that there is a ring isomorphism*

$$\gamma : H^*(X', \mathbb{Z}) \xrightarrow{\cong} H^*(X, \mathbb{Z})$$

Then X' is not projective.

The idea of the proof is to derive, from γ and the structure of the cohomology of X, the impossibility to realize over $H^1(X', \mathbb{C})$ a polarization induced by a rational 2-cohology class. By Kodaira Theorem (1.5.9), the existence of such a class is equivalent to the projectivity of the manifold and consequently, it forces any cohomology group to be endoved with a rational polarization.
To show that, we will study the Albanese torus of both X and X'. We shall use γ, to relate the cohomological structures of the manifolds and the cohomological constraints imposed by the construction of X, via blow-up of subtori.
The key fact is given by the categorical equivalence between Hodge structure of weight 1 and complex tori. Roughly speaking, there is a 1-1 corrispondence between Hodge

structure of weight 1 and complex tori; moreover this is true even for (integrally) polarized Hodge structures of weight 1 and abelian varieties (see [V02, I, Ch. 7, § 7.2.2]). Such an equivalence has already been highlightened in Remark (2.1.1).
All this is classically realized via the Picard torus associated to weight 1 Hodge structure M, defined over $\mathbb{Z}$, $M_{\mathbb{C}} = M^{1,0} \oplus M^{0,1}$,

$$Pic^0(M) = M_{\mathbb{C}}/(M^{1,0} \oplus M).$$

In the case of a Kähler manifold T

$$Pic^0(T) := Pic^0(H^1(T,\mathbb{Z})) = H^1(T,\mathbb{C})/(H^{1,0}(T) \oplus H^1(T,\mathbb{Z})).$$

Recall also the definition of the Albanese torus of a Kähler complex variety T as

$$\mathrm{Alb}\, T = H^0(X,\Omega_T)^*/H_1(T,\mathbb{Z}).$$

In addition, we will need the following classical

Proposition 2.1.8. *Let T be a projective manifold, then $Pic^0(T)$ is an abelian variety.*

Proof. [V02, I, Ch. 7, Prop. 7.16] □

Proof of Theorem (2.1.7). We immediately note that X and X' are both $2n$-dimensional. This is a consequence of the isomorphism γ and of the assumption that X' is a compact complex manifold (hence orientable), which implies that, if $\dim_{\mathbb{C}} X' = k$, then $H^{2k}(X',\mathbb{Z}) \cong \mathbb{Z}$ and for all $h > 2k$, $H^h(X') = 0$.
Consider now the Albanese tori of X and X':

$$\mathrm{Alb}\, X, \ \ \mathrm{Alb}\, X'.$$

Since $H^1(X,\mathbb{Z}) \cong H^1(X',\mathbb{Z})$, they have the same complex dimension

$$\dim_{\mathbb{C}} \mathrm{Alb}\ X = \dim_{\mathbb{C}} \mathrm{Alb}\ X' = \mathrm{rank}\ H^1(X,\mathbb{Z})/2,$$

and there are associated Albanese maps

$$\mathrm{alb}_X : X \to \mathrm{Alb}\ X,\ \mathrm{alb}_{X'} : X' \to \mathrm{Alb}\ X',$$

defined by integrating holomorphic forms along paths:

$$X \ni x \mapsto \int_{x_0}^{x} \in H^0(X,\Omega_X)^*,$$

x_0 being a choosen base point in X and the same for X'.
Now, alb_X and $alb_{X'}$ give pull-back maps in cohomology:

$$alb_X^* : H^*(\mathrm{Alb}\ X,\mathbb{Z}) \to H^*(X,\mathbb{Z}),\ alb_{X'}^* : H^*(\mathrm{Alb}\ X',\mathbb{Z}) \to H^*(X',\mathbb{Z})$$

which induce isomorphisms

$$alb_X^* : H^1(\text{Alb } X, \mathbb{Z}) \to H^1(X, \mathbb{Z}),\ alb_{X'}^1 : H^*(\text{Alb } X', \mathbb{Z}) \to H^1(X', \mathbb{Z})$$

because, by the definition of Alb X, $H^1(\text{Alb } X, \mathbb{Z}) \cong H_1(X, \mathbb{Z})^* = H^1(X, \mathbb{Z})$.
Since $H^*(\text{Alb } X, \mathbb{Z}) \cong \bigwedge^* H^1(\text{Alb } X, \mathbb{Z})$ and pull-back maps are compatible with cup-product, we identify

$$\text{alb}_X^* : H^*(\text{Alb } X, \mathbb{Z}) \to H^*(X, \mathbb{Z}) \tag{2.6}$$

with the natural map induced by cup-product

$$\bigwedge^* H^1(X, \mathbb{Z}) \to H^*(X, \mathbb{Z}),$$

and similarly for X'.
From that, we deduce that both alb_X and $\text{alb}_{X'}$ are birational maps. This follows from the commutative diagram

$$\begin{array}{ccc} H^{4n}(\text{Alb } X, \mathbb{Z}) & \xrightarrow{\text{alb}_X^*} & H^{4n}(X, \mathbb{Z}) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ H^{4n}(\text{Alb } X', \mathbb{Z}) & \xrightarrow{\text{alb}_{X'}^*} & H^{4n}(X', \mathbb{Z}) \end{array} \tag{2.7}$$

and the fact that the map alb_X is simply (up to translation) the blow-down map $X \xrightarrow{\pi} T$. By the universal property of the Albanese torus of X (see [V02, I, Ch. 12, Thm. 12.15]), there is a map $\phi : \text{Alb } X \to T$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\text{alb}_X} & \text{Alb } X \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \phi} & \\ T & & \end{array}$$

commutes. The blow-down map π is generically 1-1 and onto, while ϕ is a morphism of complex tori (hence the composition of a homomorphism of Lie groups and a translation for the group law), so that we deduce that ϕ is an isomorphism. This means that the map alb_X^* in (2.7) is an isomorphism and the same is true for $\gamma \circ \text{alb}_X^*$. Now, $\gamma : H^{4n}(\text{Alb } X, \mathbb{Z}) \to H^{4n}(\text{Alb } X, \mathbb{Z})$ is an isomorphism, forcing $\text{alb}_{X'}^* : H^{4n}(\text{Alb } X', \mathbb{Z}) \to H^{4n}(X', \mathbb{Z})$ to be an isomorphism (for the commutativity of (2.7)), hence $\text{alb}_{X'}$ is birational.
As $\text{alb}_{X'}$ is birational, it induces an isomorphism

$$\text{alb}_{X'}^* : H^0(\text{Alb } X', \Omega^2_{\text{Alb } X'}) \cong H^0(X', \Omega^2_{X'}) \tag{2.8}$$

since birational maps between complex manifolds give isomorphisms on the global holomorphic sections of exterior powers of cotangent bundle.
From (2.8), we deduce that

$$\text{alb}_{X'}^* : H^2(\text{Alb } X', \mathbb{Z}) \to H^2(X', \mathbb{Z})$$

is an isomorphism on $H^{2,0}$ and $Ker\ \mathrm{alb}_{X'*}$ has no $(2,0)$-part, i.e. it is composed only of $(1,1)$-type classes.
We claim that $Ker\ \mathrm{alb}_{X'*}$ is the image under γ^{-1} of the subgroup $\ker \mathrm{alb}_{X*}$. Indeed, Poincaré duality is given by cup-product, under the identification given by the choice of an orientation

$$H^{4n}(\cdot, \mathbb{Z}) \cong \mathbb{Z}$$

for all the considered manifolds. As γ is compatible with the cup-product, it is compatible up to sign with Poincaré duality. Moreover, γ identifies the images of the pull-back maps (2.6) for X and X', exactly as in (2.7). Since $\ker \mathrm{alb}_{X'*}$ in degree 2 is the orthogonal complement with respect to Poincaré duality of the image of the pull-back maps in degree $4n-2$, and similarly for X, the assertion follows.
Because of the previous claim, we have that $\ker alb_{X'*}$ is given by a set of integral Hodge classes of degree 2, $\gamma^{-1}(\alpha)$, $\alpha \in \ker \mathrm{alb}_{X*}$. As X is obtained from $T \times T$ by a sequence of blow-ups, the generators of this free subgroup are the classes of exceptional divisors

$$\Delta_{x_i},\ \Delta_{y_j},\ \Delta_{T\times\{0\}},\ \Delta_{\{0\}\times T},\ \Delta_{\mathrm{diag}},\ \Delta_{\mathrm{graph}},$$

over the corresponding center of blow-up x_i, y_j, $T\times\{0\}$, $\{0\}\times T$, T_{diag}, T_{graph}.
Let us denote $\delta_{\cdot} := \gamma^{-1}([\Delta_{\cdot}])$.
To each of these classes we can associate the cup-product action on the cohomology of X'; since these are Hodge classes, their action is via morphism of Hodge structure of bidegree $(1,1)$, hence the kernel is a sub-Hodge structure of the cohomology ring. In particular, we will consider the action on $H^1(X', \mathbb{Z})$. As γ is compatible with cup-product, these subgroups are the images under γ^{-1} of the subgroups

$$Ker\big\{\cup [\Delta_{\cdot}] : H^1(X,\mathbb{Z}) \to H^3(X,\mathbb{Z})\big\}. \tag{2.9}$$

We are interested in finding out these kernels.
First of all, notice that the blow-up over the points x_i, y_j may not be considered since it does not affect H^1, or even H^3. Hence, we can reduce to the case where X is obtained by blowing-up in $T\times T$ the subtorus whose proper transform has cohomology class $[\Delta_{\cdot}]$ in (2.9). From the proof of Theorem (1.4.8) and Remark (1.4.10), we know that the kernel of

$$\cup[\Delta_{\cdot}] : H^1(X,\mathbb{Z}) \to H^3(X,\mathbb{Z})$$

is equal to the kernel of the pull-back map $i_{\cdot}^*$ induced by the inclusion of the center of the exceptional divisor in $T \times T$.
In Lemma (2.1.6), we described explicitly these subspaces. These are:

1. $\ker pr_1 = \{(0,\alpha)\ \mid \alpha \in H^1(T,\mathbb{Z})\}$;
2. $\ker pr_2 = \{(\alpha,0)\ \mid \alpha \in H^1(T,\mathbb{Z})\}$;
3. $\ker pr_1 + pr_2 = \{(\alpha,-\alpha)\ \mid \alpha \in H^1(T,\mathbb{Z})\}$;
4. $\ker pr_1 + \phi_T^* \circ pr_2 = \{(-\phi_T^*(\alpha),\alpha)\ \mid \alpha \in H^1(T,\mathbb{Z})\}$.

Going back to X', we see, by what we have just established, that $H^1(X',\mathbb{Z})$ contains four integral sub-Hodge structures L_i, $i=1,\dots,4$ which are images via γ^{-1} of the soubgroups

$$Ker\, pr_1,\ Ker\, pr_2,\ Ker\, pr_1+pr_2,\ Kerpr_1+\phi_T^*\circ pr_2. \tag{2.10}$$

Now, let us consider the Picard variety of X' (which is the dual torus of the Albanese variety): $Pic^0(X')$. Given an integral sub-Hodge structure of $H^1(X',\mathbb{Z})$, this immediately gives a subtorus of $Pic^0(X')$: namely, if $L\subset H^1(X',\mathbb{Z})$ is a primitive sublattice[2] such that

$$L_{\mathbb{C}}=L^{1,0}\oplus L^{0,1}$$

where $L_{\mathbb{C}}=L\otimes\mathbb{C}$ and $L^{i,1-i}=L_{\mathbb{C}}\cap H^{i,1-i}$, $i=0,1$. Then, we define

$$T_L=L_{\mathbb{C}}/(L^{1,0}\oplus L),$$

which is clearly a complex subtorus of $Pic^0(X')$.
We get four subtori T_{L_i}, $i=1,\dots,4$ of $Pic^0(X')$, corresponding to the image via γ^{-1} of the sub-Hodge structures in (2.10). The T_{L_i}'s satisfy the following conditions, induced by the mutual position of the structures (2.10) in $H^1(X,\mathbb{Z})$:

1. $T_{L_1}\oplus T_{L_2}=Pic^0(X')$;

2. T_{L_3} is isomorphic to T_{L_1} and T_{L_2}; hence $Pic^0(X')\cong T'\oplus T'$, for a given T' isomorphic to T_{L_i}, $i=1,2$;

3. T_{L_4} is isomorphic to T'. Hence it is the transpose of the graph of an endomorphism of $\phi'_{T'}$ of T', i.e. a subset of the form $\{(-\phi'_{T'}(\alpha),\alpha),\alpha\in T'\}$.

We observe that the action of $\phi'_{T'}$ on the homology of $H_1(T',\mathbb{Z})$ is determined by the positions of the four sublattices L_i. By (2.10), via γ we can identify $\phi'_{T'*}$ to ϕ_T^*, i.e. to the dual of our initial endomorphism ϕ.
Finally, we proved that the Picard torus of X' is a product $T'\times T'$, where T' is a complex torus admitting an endomorphism which acts on $H_1(T',\mathbb{Z})$ as ϕ^*. Hence we can apply Lemma (2.1.3) and Remark (2.1.4) and deduce that T' cannot be projective. But if X' were projective, the same would be true for its Picard torus as stated in Proposition (2.1.8). So X' is not projective.

■

Remark 2.1.9. The hypothesis that the isomorphism γ was defined on the integral cohomology was necessary to prove that the sub-Hodge structure $Ker\ alb_{X'*}$ was composed exclusively by cohomology classes of type $(1,1)$.
If we had supposed thet the isomorphism γ was defined on the rational (resp. complex) cohomolgy algebra, we could not have proved that the map $alb_{X'}$ was birational (i.e. of degree 1), which implied the existence of the isomorphism $H^0(X',\Omega^2)\cong H^0(Alb\ X',\Omega^2_{Alb\ X'})$.

[2]Given a lattice M and a sublattice $L\subset M$, L is a primitive sublattice of M if M/L is a free $\mathbb{Z}$-module.

The manifold X, given above, may have any even dimensione n, for $n \geq 4$. To complete the proof of the theorem stated in the introduction we have to deal with odd values of n. But this follows from what we have just seen.

Proposition 2.1.10. *Let X be as above, and let F be an elliptic curve. Let X' be a compact Kähler manifold and assume that there is a ring isomorphism*

$$\gamma : H^*(X', \mathbb{Z}) \xrightarrow{\cong} H^*(X \times F, \mathbb{Z}).$$

Then X' is not projective.

Proof. Reproducing the arguments in the previous proof, we see immediately that the Albanese map

$$alb_{X'} : X' \to Alb(X')$$

is birational and that the kernel of the map

$$alb_{X'*} : H^2(X', \mathbb{Z}) \to H^2(AlbX', \mathbb{Z})$$

is made of Hodge classes and is equal to

$$\gamma^{-1}(Ker(alb_{X\times F*} : H^2(X \times F, \mathbb{Z}) \to H^2(Alb(X \times F), \mathbb{Z}))).$$

The group $Ker(alb_{X\times F*} : H^2(X \times F, \mathbb{Z}) \to H^2(Alb(X \times F), \mathbb{Z}))$ is generated by the classes of the exceptional divisors of the blowing down map

$$\tau : X \times F \to T \times T \times F.$$

This time, the centers of the exceptional divisors are either of the form $point \times F$ or are proper transform of subtori isomoprhic to $T \times F$. For any exceptional divisor Δ of the form $point \times F$, its cohomology class $[\Delta] \in H^2(X, \times F, \mathbb{Z})$ induces a morphism of Hodge structure

$$\cup[\Delta] : H^1(X \times F, \mathbb{Z}) \to H^3(X \times F, \mathbb{Z})$$

whose kernel, see equation (1.10), is

$$H^1(T \times T, \mathbb{Z}) \subset H^1(T \times T \times F, \mathbb{Z})$$

which we can also identify with

$$Ker\{i^* : H^1(T \times T \times F, \mathbb{Z}) \to H^1(point \times F, \mathbb{Z})\},$$

where i^* is the pull back of the inclusion map $i : point \times F \to T \times T \times F$.
Let $L \subset H^1(X', \mathbb{Z})$, $L := \gamma^{-1}(Ker(i^* : H^1(T \times T \times F, \mathbb{Z}) \to H^1(point \times F, \mathbb{Z})))$, then $L = Ker(\cup\delta)$, where $\delta = \gamma^{-1}([\Delta])$, since γ is compatible with cup-product. Being δ a Hodge class, L is a sub-Hodge structure of $H^1(X', \mathbb{Z})$ to it which corresponds a complex subtorus T_L of $Pic^0(X')$. It is clear that this construction is independent from the point considered, i.e. from the class $[\Delta]$ of exceptional divisor over a center of the form

point $\times F$ chosen.
Now we consider the four exceptional divisor Δ_i over the proper transforms of

$$0 \times T \times F,\ T \times 0 \times F,\ T_{\text{diag}} \times F,\ T_{\text{graph}} \times F.$$

There are 4 corresponding sub-Hodge structures $L_i \subset H^1(X', \mathbb{Z})$ which are the kernels of the maps

$$\cup\delta_i : H^1(X', \mathbb{Z}) \to H^3(X', \mathbb{Z}),\ \delta_i = \gamma^{-1}(\Delta_i).$$

Each of these sub-Hodge structures, L_i, provides a subtorus $T_{L_i} \subset T_L$. As above, the following relations hold:

1. $T_{L_1} \oplus T_{L_2} = T_L$;
2. T_{L_3} is isomorphic to T_{L_1} and T_{L_2}; hence $Pic^0(X') \cong T' \oplus T'$, for a given T' isomorphic to T_{L_i}, $i = 1, 2$;
3. T_{L_4} is isomorphic to T'. Hence it is the transpose of the graph of an endomorphism of $\phi'_{T'}$ of T'.

Since the action of the endomorphism $\phi'_{T'}$ on its homology groups H_1 is determined by the positions of the four sublattices L_i, we see that it has to be identified to the dual of ϕ.
Finally, T_L is a product $T' \times T'$, where T' admits an endomorphism which acts on its homology as the dual of ϕ. Hence the subtorus T_L of $Pic^0(X')$ cannot be projective, so $Pic^0(X')$ is not projective (otherwise any subtorus would be an abelian variety) and X' is not projective. □

2.2 How to change the field of definition of γ

In Theorem (2.1.7), as noticed in Remark 2.1.9, the hypothesis that γ was defined on the integer was necessary to prove the theorem, through the analysis of the Albanese varieties of X and X'. What we want to do now is trying to understand if we can change the coefficients, in order to establish different techniques.
By developing a very useful tool (Deligne's Lemma (2.2.2)), we will succed in our attempt, providing examples both in the case of rational and complex coefficients.
We first start with the case of rational coefficients.

Theorem 2.2.1. *Let X be as in Theorem (2.1.7). If X' is a Kähler manifold such that there exists a graded isomorphism of rational cohomology rings*

$$\gamma : H^*(X', \mathbb{Q}) \xrightarrow{\cong} H^*(X, \mathbb{Q})$$

then X' is not a complex projective manifold.

2.2.1 Deligne's Lemma and appplications

Let $A^* = \oplus_j A^j$ be the rational cohomology ring of a Kähler compact manifold and let $A^*_{\mathbb{C}} := A^* \otimes \mathbb{C}$ be its complexification. Let $Z \subset A^k_{\mathbb{C}}$, here $k \in \mathbb{N}$ is fixed, be an algebraic subset which is defined by homogenous equations expressed only using the ring structure of A^* (i.e. addition and cup-product).
The examples we shall consider here and in the next chapter have form:

1. $Z = \{\alpha \in A^k_{\mathbb{C}} | \alpha^l = 0 \in A^{kl}_{\mathbb{C}}\}$, where l is a given integer.
2. $Z = \{\alpha \in A^k_{\mathbb{C}} | \cup \alpha : A^l_{\mathbb{C}} \to A^{k+l}_{\mathbb{C}}$ is not injective$\}$, where l is a given integer.

We want to find conditions assuring us that the vector subspace generated by sets of this form are sub-Hodge structures of the relative cohomology group.

Lemma 2.2.2 (Deligne)**.** *Let Z be as above, and let Z_1 be an irreducible component of Z.*
Assume the $\mathbb{C}$-vector space $< Z_1 >$ generated by Z_1 is defined over $\mathbb{Q}$, i.e. $< Z_1 >= B^k_{\mathbb{Q}} \otimes \mathbb{C}$, for some $B^k_{\mathbb{Q}} \subset A^k_{\mathbb{Q}}$. Then $B^k_{\mathbb{Q}}$ is a rational sub-Hodge structure of $A^k_{\mathbb{Q}}$.

Proof. We will show that $B^k_{\mathbb{C}} =< Z_1 >$ is stable under the Hodge decomposition of $A^k_{\mathbb{C}}$, i.e. that $B^k_{\mathbb{C}} \cap \overline{(A^k_{\mathbb{C}})^{p,k-p}} = B^k_{\mathbb{C}} \cap (A^k_{\mathbb{C}})^{k-p,p}$, $\forall p \in \{0, \dots, k\}$.
Recall, that we can think the Hodge decomposition as the continous action of $\mathbb{C}^*$ on $A^*_{\mathbb{C}}$ given by

$$z \cdot \alpha = z^p \bar{z}^q \alpha, \ \alpha \in A^{p,q}.$$

So it suffices to show that $< Z_1 >$ is stable under this $\mathbb{C}^*$-action.
Notice that the action is compatible with the cup-product:

$$\forall \alpha, \beta \in A^*, \ z \cdot (\alpha \cup \beta) = z \cdot \alpha \cup z \cdot \beta,$$

hence Z, being defined only in terms of the ring structure given by the addition and the cup-product, is stable.
Now, since Z_1 is an irreducible component of Z, it must be stable for this action, too. Indeed, the $\mathbb{C}^*$-action is C^∞ for the standard euclidean structure on $A^k_{\mathbb{C}}$ (considered as a vector space) and $\mathbb{C}^*$ is a connected group (for both the Zariski topology and the euclidean one). In addition, we have that, restricting to the smooth locus of Z, this is the disjoint union of the smooth loci of its irreducible components. In particular, the smooth locus of Z_1 (which is dense in Z_1) is a connected component of this set, and can be considered as a complex manifold (hence also as a real C^∞ manifold); since $\mathbb{C}^*$ is connected and the action is C^∞, it must be stable and the same is true (by a density argument) for Z_1. □

Proof of Theorem (2.2.1). We will follow the proof of theorem (2.1.7).
Let P be the subspace of $H^2(X', \mathbb{Q})$ defined as the orthogonal complement with respect to Poincaré duality of $\bigwedge^{4n-2} H^1(X', \mathbb{Q}) \subset H^{4n-2}(X', \mathbb{Q})$. In the preceding proof we used the integral coefficients to show that P was generated by Hodge classes. Let now $P_0 \subset P$ be defined as

$$P_0 = \{\alpha \in P \mid \cup\,\alpha : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q}) \text{ is zero}\}.$$

By Lemma (2.2.2), this is a sub-Hodge structure of P. Via γ^{-1}, we can identify it with the subspace generated by the classes of exceptional divisors over points.
For P_0 is a sub-Hodge structure, we can consider the quotient P/P_0 which naturally comes with a Hodge structure induced by that of P. Let $\alpha \in P/P_0$ and consider the induced cup-product

$$\cup\alpha : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q}).$$

By extending scalar multiplication, we now introduce the algebraic subset of $P/P_0 \otimes \mathbb{C}$:

$$Z = \{\alpha \in (P/P_0) \otimes \mathbb{C} \mid \cup\,\alpha : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q}) \text{ is not injective}\}.$$

The description made of the subgroups listed in (2.10) shows that Z is the image via γ^{-1} of the union of four 1-dimensional $\mathbb{C}$-vector spaces, actually defined over $\mathbb{Q}$, generated by the classes of the four exceptional divisors over the blown-up complex subtori of $T \times T$. Applying Delgine's Lemma (2.2.2), we conclude that the four classes $\delta.$ in the proof of Theorem (2.1.7), projected in P/P_0 are Hodge classes.
Thus, considering the kernel of the maps

$$\cup\delta. : H^1(X', \mathbb{Q}) \to H^3(X', \mathbb{Q})$$

we find four sub-Hodge structures of $H^1(X', \mathbb{Q})$ whose relative positions are the same as those indicated in the proof of Theorem (2.1.7), thus we conclude analogously.

■

We now want to give another example, using complex coefficients. For this purpose, we will slightly modify the construction of X.
Start with the previous X. The four exceptional divisors $\Delta_{.}$ dominating the four complex subtori $T\times\{0\}$, $\{0\}\times T$, T_{diag}, T_{graph} are of the form $\widetilde{T}\times\mathbf{P}^{n-1}$, where $\widetilde{T}$ is obtained by blowing-up finitely many points on T^3. Indeed, the normal bundle N of each of these subtori is the trivial bundle, since its dual, the conormal bundle, is trivial, being generated by the restriction of those global holomorphic 1-form whose pull-back vanishes. Moreover, the restriction of the normal bundle to the exceptional part of $\widetilde{T}_{.}$ is trivial.
Let us blow-up one subvariety of the form $\widetilde{T}_{\{0\}\times T}\times\{\alpha_1\}\subset\Delta_{\{0\}\times T}$, then two subvarieties of the form $\widetilde{T}_{\text{diag}}\times\{\beta_1\}$, $\widetilde{T}_{\text{diag}}\times\{\beta_2\}\subset\Delta_{\text{diag}}$ and three subvarieties of the form $\widetilde{T}_{\text{graph}}\times\{\gamma_1\}$, $\widetilde{T}_{\text{graph}}\times\{\gamma_2\}$, $\widetilde{T}_{\text{graph}}\times\{\gamma_3\}\subset\Delta_{\text{graph}}$.
We have now a new smooth compact Kähler manifold, X_1, which allows us to weaken the hypothesis on the field of definition for γ.

Theorem 2.2.3. *If X_1' is a Kähler manifold such that there exists a graded isomorphism of complex cohomology algebras*

$$\gamma : H^*(X_1',\mathbb{C}) \xrightarrow{\cong} H^*(X_1,\mathbb{C}),$$

then X_1' is not a complex projective manifold.

Proof. We use the same notations as before.
Consider the cup-product multiplication map

$$\cup\alpha : H^1(X_1',\mathbb{Q})\to H^3(X_1',\mathbb{Q})$$

for $\alpha\in P/P_0$ (P_0 is again a sub-Hodge structure, by Deligne's lemma, since it is a subspace of P defined over $\mathbb{Q}$).
We introduce the algebraic subset $Z\subset P/P_0\otimes\mathbb{C}$ of those α for which $\cup\alpha$ is not injective. This algebraic subset is defined over $\mathbb{Q}$ (since, if we think of $\cup\alpha$ as linear operator, it is defined by the vanishing of the determinant) and using γ we see that it consists of the union of four complex subspaces (which intersect mutually transversally) of respective dimensions $1,2,3,4$. This can be viewed as follows: by the construction of X_1 we have that $\gamma^{-1}(P/P_0)$ is the subspace of $H^2(X_1,\mathbb{C})$ given by the class of exceptional divisors whose center is not over points, i.e. by the classes

$$\Delta_{T\times\{0\}},\ \Delta_{\{0\}\times T},\ \Delta_{\text{diag}},\ \Delta_{\text{graph}},\ \Delta_{\widetilde{T}_{\{0\}\times T}\times\{\alpha_1\}},$$
$$\Delta_{\widetilde{T}_{\text{diag}}\times\{\beta_1\}},\ \Delta_{\widetilde{T}_{\text{diag}}\times\{\beta_2\}},\ \Delta_{\widetilde{T}_{\text{graph}}\times\{\gamma_1\}},\ \Delta_{\widetilde{T}_{\text{graph}}\times\{\gamma_2\}},\ \Delta_{\widetilde{T}_{\text{graph}}\times\{\gamma_3\}}.$$

Using formula (1.10), it can be shown that the algebraic subset of classes inducing morphisms of Hodge strucutres whose kernels are non trivial is given by the following linear subspaces

$$<[\Delta_{T\times\{0\}}]>,\ <[\Delta_{\{0\}\times T}],[\Delta_{\widetilde{T}_{\{0\}\times T}\times\{\alpha_1\}}]>,\ <[\Delta_{\text{diag}}],[\Delta_{\widetilde{T}_{\text{diag}}\times\{\beta_i\}}],\ i=1,2>,$$
$$<[\Delta_{\text{graph}}],[\Delta_{\widetilde{T}_{\text{graph}}\times\{\gamma_j\}}],\ j=1,2,3>.$$

[3] Clearly, the number of points blown-up depends on which torus we are considering.

Since Z is defined over $\mathbb{Q}$ and is the union of vector subspaces of different dimensions, it follows that each of the subspaces is defined itself over the rationals. This is a consequence of the following principle, which is not difficult to prove:

Let $\mathbb{C}^n$ be the n-dimensional affine space over the complex numbers. Then a subspace $V \subset \mathbb{C}^n$ is stable for the action of $Gal(\mathbb{C}/\mathbb{Q})$ if and only if it is defined over $\mathbb{Q}$, that is, there is a base of V of vectors whose coordinates are rationals (which is equivalent to the fact that V is of the form $B_{\mathbb{Q}} \otimes \mathbb{C}$ for a suitable subspace $B_{\mathbb{C}} \subset \mathbb{Q}^n$).

In view of these facts, we can apply Deligne's Lemma, establishing that these subspaces are sub-Hodge structures of P/P_0. For each of this subspaces V, there is only one common kernel for the map $\cup\alpha$, $\alpha \in V$, which is a sub-Hodge strucutre of $H^1(X_1', \mathbb{Q})$. Again, these are exactly four sub-Hodge structure of $H^1(X_1', \mathbb{Q})$ whose mutual positions are the same as in previous proofs, via γ. In summary, $H^1(X_1', \mathbb{Q})$ splits into the sum of two copies of a rational Hodge structure admitting an automorphism which is conjugate to ϕ^t. Hence we can conclude as in the proof of Theorem (2.1.7). □

2.2.2 Simply connected examples

The aim of this section is to provide a simply connected esampe of compact Kähler manifold not having the omotopy type of a projective complex manifold.
Notice that the preceding examples made a systematic use of the Albanese variety of the complex torus T, hence we could not consider the case of simply connected manifolds.
Start again with a torus T as in section (2.1), endowed with an endomorphism ϕ_T. We will make a further assumption, namely that $\dim_{\mathbb{C}} T \geq 3$. We introduce the so-called generalized Kummer variety

$$K := \widetilde{T/\pm Id},$$

i.e. the desingularization of the quotient of T by the $-Id_T$ involution; this is obtained in the following way: obviously, the locus where the action is free (i.e. $T \setminus \{$2-torsion points$\}$) becomes a smooth manifold; thus, we have only to study what happens in a neighbourhood of the 2-torsion points. By the group law of the torus, we can just study the case of the point 0. In a neighbourhood $\tilde{U}$ of the image of 0 in $T/\pm Id$ we can give analytic coordinates by taking germs of holomorphic functions defined at 0 on T and invariant for the involution: given linear coordinates $z_1, \ldots, z_n$ in $U \subset T$, this is simply the subalgebra generated by the functions

$$z_1^2, z_2^2, \ldots, z_n^2, z_1z_2, z_1z_3, \ldots, z_1z_n, z_2z_3, \ldots, z_2z_n, \ldots, z_iz_{i+1}, \ldots, z_iz_n, \ldots z_{n-1}z_n.$$

Hence we can take $\tilde{U}$ to be a neighbourhood of the origin in the cone given by the image of the morphism

$$j_2 : \mathbb{C}^n \to \mathbb{C}^N,\ N = \binom{n+3}{n+1},$$
$$(z_1, \ldots, z_n) \mapsto (z_1^2, \ldots, z_n^2, z_1z_2, \ldots, z_1z_n, \ldots, z_iz_{i+1}, \ldots, z_iz_n, \ldots, z_{n-1}z_n).$$

Blowing up this cone at 0, we obtain the tautological bundle over the so-called 2-uple embedding of $\mathbf{P}^{n-1}$ in $\mathbf{P}^{N-1}$, which is clearly smooth. Hence we have desingularized $T/\pm Id$.
The cohomology of $T/\pm Id$ is given, by a well known theorem, by those classes (in Betti cohomology) invariant for the automorphism acting on T. By the description of the cohomology of the torus given in Remark (1.1.5), we see immediately that

$$\begin{aligned} H^i(T/\pm Id, \mathbb{C}) &= 0, && \text{for } i \text{ odd} \\ H^i(T/\pm Id, \mathbb{C}) &= H^i(T, \mathbb{C}), && \text{for } i \text{ even.} \end{aligned}$$

The desingularization map $\phi : K \to T \pm Id$, from a topological point of view, is the contraction of the exceptional divisors over 2-torsion points. Thus, we have long exact sequence in (reduced) cohomology,

$$\begin{array}{ccccccc} \cdots \longrightarrow \oplus_j \widetilde{H}^{i-1}(\mathbf{P}^{n-1}, \mathbb{C}) & \longrightarrow & \widetilde{H}^i(T/\pm Id, \mathbb{C}) & \longrightarrow & \widetilde{H}^i(K, \mathbb{C}) \\ & & & & \downarrow \\ \cdots \longleftarrow & & \widetilde{H}^{i+1}(T/\pm Id, \mathbb{C}) & \longleftarrow & \oplus_j \widetilde{H}^i(\mathbf{P}^{n-1}, \mathbb{C}) \end{array} \tag{2.11}$$

Since for odd j, $\widetilde{H}^j(T/\pm Id, \mathbb{C}) = \widetilde{H}^j(\mathbf{P}^{n-1}, \mathbb{C}) = 0$, we have the following full description of the coholomology of K

$$\begin{aligned} H^0(K, \mathbb{C}) &= \mathbb{C} \\ H^{2i+1}(K, \mathbb{C}) &= 0,\ n \in \mathbb{N} \\ H^{2i}(K, \mathbb{C}) &= H^{2i}(T, \mathbb{C}) \oplus (\oplus_j H^{2i}(\mathbf{P}^{n-1}, \mathbb{C})),\ n \in \mathbb{N}. \end{aligned}$$

Remark 2.2.4. All we have just stated is independent by the choice of the coefficients in the cohomology groups.

Furthermore, it can be shown that K is also Kähler and simply connected (see [Vln, § 3.1.2]).
The existence of ϕ on T provides a meromorphic map $\phi_K : K \dashrightarrow K$ but it is not sufficient to guarantee that it is holomorphic: just consider the case of the image of a point $t \in T$, not of order 2 for the group law, such that $\phi(t)$ is a 2- torsion point; in that case, ϕ_K would have an indeterminacy point on K, corrisponding to the image of t under the quotient map. Since ϕ is given by a homomorphism of groups, blowing up the image of t in K, we see that ϕ_K becomes a holomorphic map. We conclude that ϕ_K has a well defined graph in $K \times K$ isomorphic to the blow-up of K at those points whose images are 2-torsion points.
Now, let us blow-up in $K \times K$ the diagonal and then the proper transform of the graph of ϕ_K. This will be our variety X_2. We shall denote by $\tau : X_2 \to K \times K$ the blowing-down map. By theorem (1.4.8) we can describe the cohomology of X_2 in terms of the cohomology groups of $K \times K$, of the diagonal (which is simply a copy of K) and those of the proper trasform of the graph ϕ_K under the first blow-up. Since we will particularly

focus on the 2-dimensional and 4-dimensional cohomology we will describe this two case explicitly:

$$\begin{aligned} H^2(X_2,\mathbb{Q}) &\cong H^2(T/\pm Id,\mathbb{Q})\oplus H^2(T/\pm Id,\mathbb{Q})\oplus_i [\Delta_{x_i}\times K]\cdot\mathbb{Q} \\ &\oplus_i [K\times\Delta_{x_i}]\cdot\mathbb{Q}\oplus[\Delta_{\text{diag}}]\cdot\mathbb{Q}\oplus[\Delta_{\text{graph}}]\cdot\mathbb{Q}\end{aligned}$$

where $[\Delta x_i \times K]$ (resp. $[K\times\Delta x_i]$) are the class of exceptional divisors over 2−torsion points in each copy of K and $[\Delta_{\text{diag}}]$ (resp. $[\Delta_{\text{graph}}]$) is the class of the exceptional divisor over the diagonal (resp. over the proper transform of the graph of ϕ_K);

$$\begin{aligned} H^4(X_2,\mathbb{Q}) &\cong H^4(K\times K,\mathbb{Q})\oplus(H^2(K,\mathbb{Q})\oplus[\Delta_{\text{diag}}]_{|\widetilde{K_{\text{diag}}}}\cup\mathbb{Q})\oplus \\ &\oplus (H^2(K_{\text{graph}},\mathbb{Q})\oplus[\Delta_{\text{diag}}]_{|\widetilde{K_{\text{diag}}}}\cdot\mathbb{Q})\end{aligned}$$

The simply connectedness of X_2 is a consequence of the simple connectedness of K and the fact that we are blowing up K along simply connected smooth submanifolds[4].
With partially different tecniques from those used in the previous paragraphs, we will be able to prove the following theorem, very similar to those already proved:

Theorem 2.2.5. *If X_2' is such that there exists a graded isomorphism of rational cohomoloty rings*

$$\gamma : H^*(X_2',\mathbb{Q})\xrightarrow{\cong}H^*(X_2,\mathbb{Q})$$

then X_2' is not a complex projective manifold.

Proof. The proof will be divided into several lemmas.

Lemma 2.2.6. *Consider the subspaces*

$$A_i^2 := \gamma^{-1}(\tau^*\circ pr_i^*(\bigwedge H^1(T,\mathbb{Q}))),\ i=1,2$$

of $H^2(X_2',\mathbb{Q})$. Then A_i^2 are rational sub-Hodge structures of $H^2(X_2',\mathbb{Q})$.

Proof. Let $Z'\subset H^2(X_2',\mathbb{C})$ be the algebraic subset defined as

$$Z' := \{\alpha\in H^2(X_2',\mathbb{C})\mid \alpha^2=0\}.$$

Via γ, we can identify Z' with the algebraic subset $Z\subset H^2(X_2,\mathbb{C})$ defined analogously. We observe that in $H^2(X_2,\mathbb{C})$

$$Z = Z_1\cup Z_2 \tag{2.12}$$

where

$$Z_i := \{\tau^*\circ pr_i^* \mid \alpha\in\bigwedge^2 H^1(T,\mathbb{C}),\alpha^2=0\in\bigwedge^4 H^4(T,\mathbb{C})\},\ i=1,2.$$

[4]It follows from Van Kampen's theorem

This is a consequence of the fact that X_2 has been deduced from $(T/\pm Id)\times(T/\pm Id)$ by a sequence of blow-ups with centers of codimension ≥ 3, hence the square of classes of exceptional divisors never vanishes in $H^4(X_2,\mathbb{C})$.
We derive that $< Z_i >= \tau^*\circ pr_i^*(\wedge^2 H^1(T,\mathbb{C}))$, since the latter space is generated by those classes whose square vanishes in $H^4(T,\mathbb{C})$.
Applying Deligne's Lemma (2.2.2) to $Z_i' := \gamma^{-1}(Z_i)$ we end the proof - since γ is defined over $\mathbb{Q}$. □

Let now A^* be the subalgebra of $H^*(X_2',\mathbb{Q})$ generated by $A_1^2\oplus A_2^2$ and let $P\subset H^2(X_2',\mathbb{Q})$ be the orthogonal complement with respect to Poincaré duality of A^{4n-2}. We claim that

$$P=\gamma^{-1}(E),$$

where E is the subspace of $H^2(X_2,\mathbb{Q})$ generated by the classes of exceptional divisors over $(T/\pm Id)\times(T/\pm Id)$. Once again, this is a consequence of formula (1.10) and the fact that the centers of exceptional divisors have codimension > 2, which implies that the restriction of A^{4n-2} to the centers is zero.
As in the previous section, consider the algebraic subset

$$S' := \{\alpha\in P \mid \cup\,\alpha : A^2\to H^4(X_2',\mathbb{C}) \text{ is not injective}\}.$$

S' is the image under γ^{-1} of the corresponding $S\subset E$. S is the union of four vector spaces defined over $\mathbb{Q}$, namely

$$<[\Delta_{\text{diag}}]>,\ <[\Delta_{\text{graph}}]>,\ <[\Delta_{x_i}\times K],\ i=1,\dots,2^{2n}>,\ <[K\times\Delta_{x_i}],\ i=1,\dots,2^{2n}>.$$

By Deligne's lemma, we conclude that the classes

$$\delta_{\text{diag}} := \gamma^{-1}([\Delta_{\text{diag}}]),\ \delta_{\text{graph}} := \gamma^{-1}([\Delta_{\text{graph}}])$$

are Hodge classes in $H^2(X_2',\mathbb{Q})$. Hence the kernels of the cup-product maps

$$\cup\delta.: A^2\to H^4(X_2',\mathbb{Q})$$

are rational sub-Hodge structures of $A_1^2\oplus A_2^2$.
Using γ to examine their relative positions, we determine that A_1^2 and A_2^2 are isomoprhic as rationalå Hodge structures and that such a rational Hodge structure carries an automorphism which acts as ϕ_T^* on $A_i^2,\ i=1,2$.
We can restate the content of Lemma (2.1.3) in the following form:

Assume the $\mathbb{Q}$-vector space $\bigwedge^2 H^1(T,\mathbb{Q})$ is endowed with a Hodge structure which is preserved by $\wedge^2\phi^t$. Then either this Hodge structure is trivial (i.e. it contains only Hodge classes) or it has no non-zero Hodge class.

It is readily seen that, if we want X_2' to be projective, only the second case can occur.

Lemma 2.2.7. *If the Hodge structures on A_i^2 is trivial, then X_2' cannot be projective.*

Proof. Recall that $A_i^2 = \gamma^{-1}(\tau^* \circ pr_i^*(\bigwedge H^1(T,\mathbb{Q})))$. We claim that there is a subspace $V \subset A_i^2$, $\dim_{\mathbb{C}} V \geq 2$ such that for any $\alpha \in V$, $\alpha^2 = 0 \in H^4(X_2',\mathbb{Q})$: just consider those classes in A_i^2 that are of the form $\theta = \gamma^{-1}(\tau^* \circ pr_i^*(\alpha \wedge \beta)), \beta \in H^1(T,\mathbb{Q})$ for a fixed $\alpha \in H^1(T,\mathbb{Q})$. It is clear now that the square of such classes will vanish.
If the Hodge structure of A_i^2 were trivial and X_2' projective, by the Hodge index theorem we would reach a contradiction: in fact, given a $(1,1)$ ample class $c \in H^2(X_2',\mathbb{Q})$ (this is a consequence of the assumption of projectivity for X_2') the quadratic form $q_c(\beta) := c^{2n-2}\beta^2$, $\beta \in H^2(X_2',\mathbb{Q})$, when restricted to the space of $(1,1)$ rational classes, will have only one positive sign. But for any $\alpha \in V$, $q_c(\alpha) = 0$ and $\dim V \geq 2$, as noted above, which is impossible, since in this kind of situation the maximal dimension of a subspace isotropic for q_c is one. □

The last lemma implies that the sub-Hodge A^2 does not contain any Hodge class if X_2' is projective (i.e. if there is a rational polarization on the second rational cohomology group - given by an ample class). Then any degree 2 Hodge class has to be contained in P.

Lemma 2.2.8. *For any class $c \in P$, the intersection form $q_c(\alpha) = c^{2n-2}\alpha^2$ on $H^2(X_2',\mathbb{Q})$ vanishes on A^2.*

Proof. Since P is the image, via γ, of the subspace generated by classes of exceptional divisors in X_2 and γ preserves the cup-product, it suffices to show the analogous result for X_2.This is a consequence of the following fact: the map

$$\begin{array}{ccc} H^2(T/\pm Id,\mathbb{Q})^{\otimes 2} & \rightarrow & H^4(T/\pm Id,\mathbb{Q}) \\ \alpha \otimes \beta & \mapsto & \alpha\beta \end{array}$$

contains in its image a basis of $H^4(T/\pm Id)$. Hence, via γ, q_c induces a morphism of Hodge structure

$$\begin{array}{rcl} H^4(T/\pm Id,\mathbb{Q}) & \rightarrow & H^{4n}(T/\pm Id,\mathbb{Q}) \cong \mathbb{Q}, \\ \theta & \mapsto & \gamma(c)^{2n-2}\theta. \end{array}$$

Such a morphism would give rise to a Hodge class in $H^{4n-4}(T/\pm Id,\mathbb{Q})$, but this is not possible as we have already noticed that such class can not exist. □

The proof of Theorem (2.2.5) can now be concluded by contradiction. If X_2' were projective, by Lemma (2.2.7) any degree 2 Hodge class on X_2' should be contained in P, and P should contain at least an ample class c. But then, again by the Hodge index theorem, the form q_c would not vanish on A^2, contradicting the last Lemma: in fact, $A^2 \otimes \mathbb{C}$ decomposes into a direct sum

$$A^2 \otimes \mathbb{C} = H^{2,0}(X_2') \oplus H^{0,2}(X_2') \oplus G \otimes \mathbb{C},$$

where G is a trivial (i.e. containing only $(1,1)$ classes) sub-Hodge structure of both A^2 and H^2. By the Hodge index Theorem, every polarization given by an ample class c is positive definite on $H^{2,0}$ and $H^{0,2}$ (these two being part of the primitive cohomology) and negative definite on G. This implies that q_c could not vanish on A^2 contradicting Lemma (2.2.8). □

Chapter 3

The birational Kodaira problem

There is a natural birational version of the original Kodaira problem which asks:

Question. *Let X be a compact Kähler manifold. Does there exist a bimeromorphic model X' of X (i.e. a compact complex manifold X' and a bimeromorphic map $\psi : X' \dashrightarrow X$) which deforms to a complex projective manifold?*

We shall always refer to that question with the expression birational Kodaira problem.
Counterexamples to the classical Kodaira problem given in the previous chapter naturally fullfill the hypotheses of the birational Kodaira problem. In fact they are bimeromorphic to products either of tori or of generalized Kummer manifolds and these are well known to deform to projective manifolds.
Voisin showed in [V05] that it is possible to construct Kähler compact manifolds X for which the answer to the above question is negative.
In this chapter we examine in full details these manifolds: more precisely, we show that

Theorem. *In any even dimension, greater than 10, there are Kähler compact manifolds providing counterexamples to the birational Kodaira problem, i.e. such that any of their bimeromorphic model has not the homotopy type of a projective complex manifold.*

In Section 3.1, we illustrate the basic construction of such counterexamples, starting from the manifolds considered in Chapter 2.
In Section 3.2, we analyze the geometric structures of these manifolds and we show some results on the Hodge strucutres of certain cohomology algebras.
Finally, Section 3.3 is devoted to the proof of the main theorem of the chapter. This theorem shows the impossibility for the manifolds constructed to provide bimeromorphic models having projective deformations, due to a Hodge-theoretic argument very similar to that illustrated in Section 2.2.2 of Chapter 2.

3.1 Construction of a counterexample

We start as in Chapter 1, considering an n-dimensional complex torus T admitting an endomorphism

$$\phi_T : T \to T$$

that satisfies property $(*)$.
We make a further assumption, namely

$$dim_{\mathbb{C}} T \geq 4.$$

Consider now the dual torus of T, $\hat{T}$. Using the same notations as in Chapter 2, $\hat{T}$ is defined in the following way:

$$\hat{T} := \Gamma^*_{\mathbb{C}}/\Gamma^* \oplus {\Gamma'}^{\perp},$$

where * indicates the dual space and ${\Gamma'}^{\perp}$ indicates the space ortogonal to Γ' in $\Gamma^*_{\mathbb{C}}$[1]. From a geometrical point of view, by the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{exp(2\pi i \cdot)} \mathcal{O}^* \longrightarrow 0$$

and the associated long exact sequence in cohomology

$$\cdots \longrightarrow H^1(T,\mathbb{Z}) \longrightarrow H^1(T,\mathcal{O}) \longrightarrow H^1(T,\mathcal{O}^*) \xrightarrow{c_1} H^2(T,\mathbb{Z}) \longrightarrow \cdots,$$

we see that $\hat{T}$ is the torus

$$Pic^0 := H^1(T,\mathbb{C})/(H^{1,0}(T) \oplus H^1(T,\mathbb{Z})),$$

i.e. the group of line bundles whose first Chern class vanishes (these are exactly the topologically trivial line bundles).
On the product $T \times \hat{T}$ there is a natural line bundle $\mathcal{P}$. $\mathcal{P}$ is called the Poincaré line bundle and it is characterized by the following properties:

- for any $t \in \hat{T}$ parameterizing a line bundle L_t on T we have

$$\mathcal{P}_{|T\times\{t\}} \cong L_t;$$

- $\mathcal{P}_{|\{0\}\times\hat{T}} \cong \mathcal{O}_{\hat{T}}$.

The fundemental result on $\mathcal{P}$ is the following:

Theorem 3.1.1. *There exists a unique Poincaré line bundle $\mathcal{P}$ on $T \times \hat{T}$ defined up to isomorphism.*

Proof. See [BL92, II, Ch. 2, Thm. 2.5.1]. □

Let us also note that the Chern class of $\mathcal{P}$, $c_1(\mathcal{P}) \in H^2(T \times \hat{T},\mathbb{Q})$ is nothing but $Id_T \in H^1(T,\mathbb{Q}) \otimes H^1(\hat{T},\mathbb{Q}) \cong End_{\mathbb{Q}}(H^1(T,\mathbb{Q}))$.
Recall that, on T, we have the endomorphism ϕ_T, so we shall also consider the line bundle

$$\mathcal{P}_\phi := (\phi, Id)^*\mathcal{P}.$$

[1]Given a vector space V and a subspace $M \subset V$, let V^* be the dual of V. The space orthogonal to M in V^*, indicated by $M^\perp$, is the set $M^\perp := \{v \in V^* \mid v(m) = 0, \forall m \in M\}$

Thus, on $T \times \hat{T}$ there are the following rank 2 vector bundles

$$E = \mathcal{P} \oplus \mathcal{P}^{-1},\ E_\phi = \mathcal{P}_\phi \oplus \mathcal{P}_\phi^{-1}$$

and their associated projective bundles $\mathbf{P}(E),\ \mathbf{P}(E_\phi)$.
We note that the commuting involutions $(-Id, Id),\ (Id, -Id)$ of $T \times \hat{T}$ lift to commuting involutions $i,\ \hat{i}$ (resp. $i_\phi,\ \hat{i}_\phi$) on E (resp. E_ϕ). In fact, there are isomorphisms

$$\begin{array}{ccc} (-Id, Id)^*\mathcal{P} \cong \mathcal{P}^{-1} & , & (Id, -Id)^*\mathcal{P} \cong \mathcal{P}^{-1} \\ (-Id, Id)^*\mathcal{P}_\phi \cong \mathcal{P}_\phi^{-1} & , & (Id, -Id)^*\mathcal{P}_\phi \cong \mathcal{P}_\phi^{-1} \end{array}$$

which can be made canonical by a choiche of trivialization

$$\mathcal{P}_{|(0,0)} \cong \mathbb{C},$$

$(0,0)$ being a fixed point of both $(Id, -Id)$ and $(-Id, Id)$.
The compact Kähler manifold we shall consider is the following:
let us start with the fibered product

$$\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi),$$

which is clearly a Kähler compact manifold. $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ admits commuting involutions

$$(i, i_\phi),\ (\hat{i}, \hat{i}_\phi)$$

lifting $(-Id, Id)$, $(Id, -Id)$ respectively. The quotient Q of $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ by the group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, generated by the liftings of the involutions, is singular along the non-free locus of the action. Anyway, this quotient admits a Kähler compact desingularization (see [V05, § 1]), i.e. a compact Kähler manifold K of dimension $2n+2$ and a map $\mu : K \to \mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$ which is 1-1 over the smooth locus of the quotient.
Clearly, in general, there are different possible choiches of a desingularization. For our purpose, we will suppose that we have fixed a Kähler desingularization of the quotient $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$ (in the rest of the chapter it will not be important which one we have fixed). We shall denote such a Kähler desingularization with X.
Note that, if K is the Kummer variety of T, as described in Chapter 2, and similarly $\hat{K}$ is the Kummer variety of $\hat{T}$, then over $K_0 \times \hat{K}_0$, X is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle. Here K_0 is the Zarisky open set $T_0/\pm Id$ of K, where

$$T_0 := T \setminus \{2-\text{torsion points}\}$$

and similarly for $\hat{K}_0$.
We want to deal with a weaker version of the Kodaira problem. More precisely, we want to show that for any compact complex manifold X' bimeromorphic to X (and to any desingularization of the quotient of $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi),\ (\hat{i}, \hat{i}_\phi) >$), X' can not deform to a complex projective manifold.

The idea is roughly the same as that explained in the previous chapter: through a detailed analysis of the rational cohomology algebra of X' and of its analytic structure, we will show that any cohomology algebra isomorphic to $H^*(X', \mathbb{Q})$ is not the cohomology algebra of a projective manifold. The exact statement is the following:

Theorem 3.1.2. *Let X' be any compact complex manifold bimeromorphically equivalent to X, and let Y be a Kähler compact manifold. Assume that there is an isomorphism of graded algebras:*

$$\gamma : H^*(Y, \mathbb{Q}) \xrightarrow{\cong} H^*(X', \mathbb{Q}).$$

Then Y is not projective.

The proof of the theorem will be given in the last section of the chapter.
Note that in Theorem (3.1.2) we do not assume that X' is a Kähler manifold. Indeed, this is not necessary since, in order to complete the analysis of the cohomology algebra of X', we only need to know that we have a Hodge decomposition on $H^*(X', \mathbb{C}) \cong H^*(X', \mathbb{Q}) \otimes \mathbb{C}$ which is functorial for pull-back maps in cohomology.
In order to do this we have to introduce the following

Definition 3.1.3. *Let S be a compact complex manifold. We say that $H^k(S, \mathbb{C})$ admits a Hodge decomposition in the strong sense if*

1. *for all p and q with $p + q = k$ the Hodge (p, q)-subspace $H^{p,q}(X)$ already defined can be identified with the subspace of $H^k(X; C)$ consisting of classes representable by closed forms of type (p, q). The resulting map $H^{p,q}(S) \to H^{p,q}_{\bar\partial}(S) \cong H^q(\Omega^p, S)$ is required to be an isomorphism.*

2. *there is a direct decomposition*

$$H^k(S, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(S).$$

3. *the natural morphism from Bott-Chern cohomology to De Rham cohomology*

$$H^{p,q}_{BC}(X) = \frac{d - \text{closed forms of type } (p, q)}{\partial\bar\partial H^0(\mathcal{E}^{p-1,q-1}, s)} \to H^{p+q}_{DR}(S, \mathbb{C})$$

which sends the class of a d-closed (p, q)-form to its De Rham class is injective with image $H^{p,q}(S)$.

The existence of such a decomposition is guaranteed by the following

Theorem 3.1.4. *Let S_1, S_2 compact complex manifolds. Suppose S_1 is Kähler and that there is a map $f : S_1 \to S_2$ holomorphic and surjective map. Then $H^*(S_2, \mathbb{Q})$ admits a Hodge decomposition in the strong sense: in fact, $f^* : H^*(S_2, \mathbb{Q}) \to H^*(S_1, \mathbb{Q})$ is injective and $f^*(H^*(S_2, \mathbb{Q}))$ is a rational sub-Hodge structure of $H^*(S_1, \mathbb{Q})$.*

Proof. [PS08, Ch. 2, Thm. 2.29] □

This is not exactly our case since we are dealing with a meromorphic map from a compact Kähler manifold, X, to a compact complex manifold, X'. But, by Hironaka's Theorem, we can resolve the indeterminacy locus of the meromorphic map on a Kähler manifold by a sequence of blow-ups on smooth compact submanifolds; since Kähler property is stable for blow-up of compact submanifolds, we can reduce ourselves to the case in which X' is dominated by a Kähler compact manifold Y (i.e. there is a surjective holomorphic morphism $\tau : Y \to X'$ which is generically 1-1 outside an analytic closed subset of Y) and apply Theorem (3.1.4).

3.2 The structure of the rational cohomology algebra of X'

Recall that in the previous section we choosed the compact Kähler manifold X to be a desingularization of the quotient $\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$. Let us note, that we have a natural map

$$\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) > \to T \times \hat{T}/ < (-Id, Id), (Id, -Id) >,$$

simply as the involutions $(i, i_\phi), (\hat{i}, \hat{i}_\phi)$ are liftings of the involutions $(-Id, Id), (Id, -Id)$ respectively. Moreover, $T \times \hat{T}/ < (-Id, Id), (Id, -Id) >= (T/ \pm Id) \times (\hat{T}/ \pm Id)$, thus, composing, we have a map

$$q : X \to (T/ \pm Id) \times (\hat{T}/ \pm Id).$$

Recall also that we have a bimeromorphic holomorphic map

$$\pi : K \times \hat{K} \to (T/ \pm Id) \times (\hat{T}/ \pm Id)$$

hence we have a meromorphic map

$$\bar{q} : X \dashrightarrow K \times \hat{K}.$$

By Hironaka's Theorem, up to a bimeromorphic morphism, we can take X to be Kähler and such that $\bar{q}$ is actually holomorphic. This will be our assumption. Hence we will suppose that X is a Kähler desingularization of $\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$ and that the map $\bar{q} : X \to K \times \hat{K}$ is holomorphic.
Here is a first consequence of such an assumption.

Lemma 3.2.1. *Let $\psi : X' \dashrightarrow X$ be any meromorphic map. Then $q \circ \psi$ is holomorphic.*

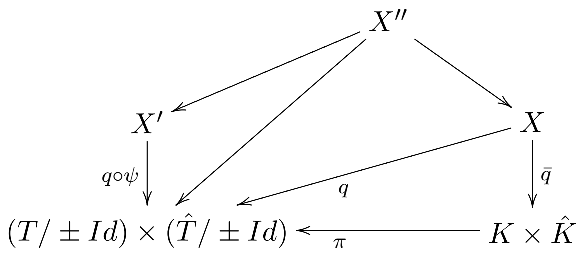
Proof. The complex manifold $T \times \hat{T}$ does not contain any closed complex curve: suppose by contradiction that there were a curve C in $T \times \hat{T}$. Then, by projecting on T and $\hat{T}$, there would be a curve C' on either T or $\hat{T}$. Such a curve would give a Hodge class $[C']$ in the degree 2 rational cohomology of either T or $\hat{T}$, but this is not possible, since we have already seen that Hodge classes do not exist in degree 2 cohomology on both T and $\hat{T}$.
It follows that the quotient $(T/ \pm Id) \times (\hat{T}/ \pm Id)$ can not contain any rational curve (otherwise its counterimage would be a curve). Hence by a well known criterion for meromorphic maps with value in a smooth manifold, $q \circ \psi$ has to be holomorphic (see [KM98, Ch.1, Cor. 1.4]). □

Since $q \circ \psi$ is holomorphic, it follows that $H^*(X', \mathbb{Q})$ contains a subalgebra

$$A^* := (q \circ \psi)^* H^*((T/ \pm Id) \times (\hat{T}/ \pm Id), \mathbb{Q})$$

isomorphic to $H^*((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q})$.
This is possible in view of the following commutative diagram

(3.1)



where X'' is Kähler, bimeromorphic to X' and the map $X'' \to X'$ is a holomorphic morphism, as already explained at the end of Section 3.1.
By the Künneth formula, there is an isomorphism

$$H^*((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q}) \cong H^*(T/\pm Id, \mathbb{Q}) \otimes H^*(\hat{T}/\pm Id, \mathbb{Q}).$$

Recall that $H^*(T/\pm Id, \mathbb{Q}) = H^{even}(T, \mathbb{Q})$ and the same is true for $\hat{T}$.
We shall denote by A_1^* (resp. A_2^*), the subalgebra $(q \circ \psi)^* H^*((T/\pm Id) \times \{0\})(resp.(q \circ \psi)^* H^*(\{0\} \times (\hat{T}/\pm Id)))$.
Now, we claim that the degree 2 cohomology of X' is generated over $\mathbb{Q}$ by the degree 2 cohomology of A^* and by degree 2 Hodge classes.
First of all, we note that this is true for X. By construction, X is a Kähler manifold bimeromorphic to $\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$. We can summarize this fact by saying that X is a smooth model of $\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)/ < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$. Now we note the following

Proposition 3.2.2. *Let V be a complex manifold of dimension n and let G be a finite group of analytic automorphisms of V. Let $\widetilde{V}$ a smooth model of V/G, then we have*

$$\dim_{\mathbb{C}} H^0(\widetilde{V}, \Omega^k) = \dim_{\mathbb{C}} H^0(V, \Omega^k)^G, \ k = 1, \ldots, n,$$

where $H^0(V, \Omega^k)^G$ is the vector space of holomorphic k-forms invariant under the action of G.

Proof. [U75, Ch. 4, Prop. 9.24] □

Holomorphic 2-forms on $\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)$ are given by pulling-back holomorphic 2-forms on $T \times \hat{T}$ under the map

$$\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi) \to T \times \hat{T}$$

induced by the fibre product. As a consequence of that and of the fact that the involutions $(i, i_\phi), (\hat{i}, \hat{i}_\phi)$ are liftings on $\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi)$ of the involutions $(-Id, Id), (Id, -Id)$ on $T \times \hat{T}$, we see that

$$\dim_{\mathbb{C}} H^0(\mathbb{P}(E) \times_{T \times \hat{T}} \mathbb{P}(E_\phi), \Omega^2)^{<(i,i_\phi),(\hat{i},\hat{i}_\phi)>} = \dim_{\mathbb{C}}(T \times \hat{T}, \Omega^2)^{<(-Id,Id),(Id,-Id)>}.$$

Following the cohomological description of the generalized Kummer variety given in Section 2.2.2 of Chapter 2, it follows that

$$\dim_{\mathbb{C}}(T \times \hat{T}, \Omega^2)^{<(-Id,Id),(Id,-Id)>} = \dim_{\mathbb{C}} H^0(K \times \hat{K}, \Omega^2).$$

Since $\bar{q}$ induces an injective homomorphism

$$\bar{q}^* : H^0(K \times \hat{K}, \Omega^2) \to H^0(X, \Omega^2),$$

we conlcude that $\bar{q}^*$ is actually an isomorphism.
As this property is invariant under bimeromorphic transformations, if it is true for X, thus it is true also for X'.

3.2.1 Some Lemmas on the Hodge structures of the $A^*_{i\mathbb{Q}}$'s

This technical subsection is used to establish some results on the cohomological structure of the subalgebras $A^*_{i\mathbb{Q}} \subset H^*(X', \mathbb{Q})$, $i = 1, 2$. All these facts will be recalled constantly in the following paragraphs: anyway, they have their own interest as they show how complicated a rational Hodge structure can be. Moreover, they also underline the importance of studying the position of the rational cohomology in terms of the Hodge decomposition.
In this particular case, difficulties arise from the existence on the $A^*_{i\mathbb{Q}}$'s of endomorphisms of Hodge structure having no stable subspace (namely $\wedge^*\phi$ and $\wedge^*\phi^t$). Such endomorphisms not only prevent the $A^*_{i\mathbb{Q}}$'s from having rational Hodge classes, but also cause many other interesting phenomena.
First of all, let us introduce the following definition, that allows us to understand how large a given Hodge structure is.

Definition 3.2.3. *Given a rational vector space H, provided with a Hodge structure of weight k, $H_{\mathbb{C}} = H \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}$, the level of H is*

$$\max_{H^{p,q} \neq 0} (p - q).$$

Remark 3.2.4. Clearly, by virtue of the Hodge symmetry $\overline{H^{p,q}} = H^{q,p}$, the level of a Hodge structure is always non-negative.
Given a Hodge structure of weight 4, H, e.g. $H^4(X', \mathbb{Q})$, such that

$$H_{\mathbb{C}} = \bigoplus_{\substack{p+q=4,\\ p \geq 0, q \geq 0}} H^{p,q}$$

it is a consequence of the preceding definition that H is of level 4 if and only if $H^{4,0} \neq 0$, otherwise the level of H will be ≤ 2.
Let us underline the fact that the level of a sum of Hodge structures is the supremum of the levels; hence, fixed a Hodge structure K of level k and a number $m \in \mathbb{N}$, we can talk of the maximal sub-Hodge structure of K of level m.

In the case of $A^4_{i\mathbb{Q}}$, $i = 1, 2$ the endomorphisms $\wedge^2\phi^t$ and $\wedge^2\phi$ cause Hodge structures to be rather chaotic, in the following sense:

Lemma 3.2.5. *$A^4_{1\mathbb{Q}}$ and $A^4_{2\mathbb{Q}}$ do not contain non trivial sub-Hodge structure of level* 2

Proof. Since $\dim_{\mathbb{C}} T \geq 4$, then $A^4_{i\mathbb{Q}}$, $i = 1, 2$ are of Hodge level 4: consider a complex basis $\{e_1, \dots, e_4, \dots, e_{4+j}\}$, $j \geq 0$ for the universal covering of T, then the 4-form $de_1 \wedge de_2 \wedge de_3 \wedge de_4$ will give a closed 4-form on T, whose cohomology class will be non-zero and of $(4, 0)$-type.
By the assumption made on T (namely, the existence of ϕ_T), we know that ϕ_T^* acts in an irreducible way on $\wedge^4 H^1(T, \mathbb{Q}) = A^4_{1\mathbb{Q}}$: in fact, let $S \subset A^4_{1\mathbb{Q}}$ be a stable subspace for $\wedge^4\phi_T^*$, then $S \otimes \mathbb{C}$ is also stable, hence it is generated by a certain set of eigenvectors for ϕ_T^*. Reasoning as in Lemma (2.1.3) of Chapter 2, we see that $S = 0$ or $S = A^4_{1\mathbb{Q}}$.
Let us indicate with $N_2A^4_{1\mathbb{Q}}$ the maximal sub-Hodge strucutre of level 2 of $A^4_{1\mathbb{Q}}$. As ϕ_T^* acts via a morphism of Hodge structure, $N_2A^4_{1\mathbb{Q}}$ will be stable for that action. By the irreducibility of ϕ, we have only two possibilities:

- $N_2A^4_{1\mathbb{Q}} = A^4_{1\mathbb{Q}}$;
- $N_2A^4_{1\mathbb{Q}} = 0$.

As $A^4_{1\mathbb{Q}}$ is of Hodge level 4, we see that $N_2A^4_{1\mathbb{Q}} = 0$. The same proof will work for $A^4_{2\mathbb{Q}}$. □

Given two rational Hodge structures L, M both of even weight, consider the tensor product $L \otimes M$ with the induced Hodge structure (as explained in Section 1.2 of Chapter 1). Suppose that M has a rational Hodge class, m, and L has a rational Hodge class, l. Then we have natural morphisms of Hodge strucutres

$$Id \otimes m : L \to L \otimes M \quad , \quad t \mapsto t \otimes m,$$
$$l \otimes Id : M \to L \otimes M \quad , \quad s \mapsto l \otimes s.$$

We know that both $A^2_{1\mathbb{Q}}$ and $A^2_{2\mathbb{Q}}$ have no rational Hodge classes. Hence it is a natural question to ask whether there might be anyway morphisms of Hodge structure

$$A^2_{i\mathbb{Q}} \to A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}},\ i = 1, 2.$$

Lemma 3.2.6. *There are no non-zero morphism of Hodge structure (of bidigree* $(1, 1)$*) from $A^2_{1\mathbb{Q}}$ or $A^2_{1\mathbb{Q}}$ to $A^2_{1\mathbb{Q}} \otimes A^2_{\mathbb{Q}}$.*

Proof. By the isomorphisms $A^*_{1\mathbb{Q}} \cong H^*(T/ \pm Id, \mathbb{Q}) \cong H^{even}(T, \mathbb{Q})$ (and similarly for $A^*_{2\mathbb{Q}}$ with $\hat{T}$ instead of T), and the fact that $\hat{T}$ is the dual torus of T, we can identify

$$A^2_{1\mathbb{Q}} = H^2(T, \mathbb{Q}) = \bigwedge^2 H^1(T, \mathbb{Q}) \cong \bigwedge^2 \Gamma^*_{\mathbb{Q}}, \tag{3.2}$$

$$A^2_{2\mathbb{Q}} = H^2(\hat{T}, \mathbb{Q}) = \bigwedge^2 H^1(\hat{T}, \mathbb{Q}) \cong \bigwedge^2 \Gamma_{\mathbb{Q}}. \tag{3.3}$$

On $T, \hat{T}$ we have automorphisms $\phi_T, \phi_{\hat{T}}$, and the induced cohomology automorphisms $\phi_T^*, \phi_{\hat{T}}^*$ on $H^2(T,\mathbb{Q}), H^2(\hat{T},\mathbb{Q})$ acting on these vector spaces like $\wedge^2\phi^t, \wedge^2\phi$ respectively. Consider now the eigenvalues of ϕ on $\Gamma_\mathbb{C}$, $\{\lambda_1,\ldots,\lambda_{2n}\}$ and let $\{e_1,\ldots,e_{2n}\}$ be a corresponding basis of eigenvectors. Let $\{e_i^*\}$ be the dual basis of $\{e_i\}$ in $\Gamma_\mathbb{C}^*$. Unless changing the ordering, $\{e_1,\ldots,e_n\}$ is a complex basis for Γ': in other words, $e_i \in H^1(\hat{T},\mathbb{C}), i \leq n$ has Hodge type $(1,0)$, while $e_j^* \in H^1(T,\mathbb{C}), j > n$ has Hodge type $(1,0)$.
We want to study morphisms of Hodge structure from $A^2_{i\mathbb{Q}}, i = 1,2$ to $A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$. We have an isomorphism of vector spaces

$$Hom_\mathbb{C}(A^2_{i\mathbb{Q}}, A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \cong A^{2*}_{i\mathbb{Q}} \otimes A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}},$$

but in view of (3.2)

$$\begin{aligned} & A^{2*}_{1\mathbb{Q}} \otimes A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}} \\ \cong & \bigwedge^2 \Gamma_\mathbb{Q} \otimes \bigwedge^2 \Gamma^*_\mathbb{Q} \otimes \bigwedge^2 \Gamma_\mathbb{Q}, \end{aligned} \tag{3.4}$$

which is naturally equipped by a weight 6 Hodge structure, that is the tensor product structure of the weight 2 structures on $\Gamma_\mathbb{Q}$ and $\Gamma^*_\mathbb{Q}$. The case of $A^{2*}_{1\mathbb{Q}} \otimes A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$ may be treated as well.
The following statement reduces our problem to the pursue of Hodge classes in $A^{2*}_{1\mathbb{Q}} \otimes A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$.

Lemma 3.2.7. *Let α be a class in*

$$A^{2*}_{1\mathbb{Q}} \otimes A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$$

representing a morphism

$$\tilde{\alpha} : A^2_{1\mathbb{Q}} \to A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}.$$

Then α is a Hodge class if and only if $\tilde{\alpha}$ is a morphism of Hodge structure.

Proof. [V02, I, Ch. 11, Lemma 11.41] □

Hence, consider the space

$$S := Hdg(\bigwedge^2 \Gamma_\mathbb{Q} \otimes \bigwedge^2 \Gamma^*_\mathbb{Q} \otimes \bigwedge^2 \Gamma_\mathbb{Q}),$$

which is stable under the action of the three commuting morphisms of Hodge structure

$$\wedge^2\phi \otimes Id \otimes Id, Id \otimes \wedge^2\phi^t \otimes Id, Id \otimes Id \otimes \wedge^2\phi.$$

By the descritpion of the action of $\wedge^2\phi$ and $\wedge^2\phi^t$, we conclude that $S_\mathbb{C} = S \otimes \mathbb{C}$ is generated by eigenvectors of these actions, i.e. element of the form

$$e_i \wedge e_j \otimes e_k^* \wedge e_l^* \otimes e_r \wedge e_s. \tag{3.5}$$

Now, for $a, b, c \in \mathbb{Z}$, consider the endomorphism

$$\Phi_{abc} := (\wedge^2\phi)^a \otimes (\wedge^2\phi^t)^b \otimes (\wedge^2\phi)^c.$$

In the basis given by elements of the form (3.5), Φ is diagonal, with corresponding eigenvalues

$$(\lambda_i\lambda_j)^a(\lambda_k\lambda_l)^b(\lambda_r\lambda_s)^c.$$

Exactly as in Lemma (2.1.3) the Galois group of the field $K = \mathbb{Q}[\lambda_1, \dots, \lambda_{2n}]$ over $\mathbb{Q}$ acts on the λ_i and shall leave stable the set E_{abc} of eigenvalues of Φ_{abc} on S, since S is defined over $\mathbb{Q}$. Remember that $Gal(K/\mathbb{Q})$ is the simmetric group on $2n$ elements S_{2n}, so that if

$$(\lambda_i\lambda_j)^a(\lambda_k\lambda_l)^b(\lambda_r\lambda_s)^c \in E_{abc}$$

then also

$$(\lambda_{\sigma(i)}\lambda_{\sigma(j)})^a(\lambda_{\sigma(k)}\lambda_{\sigma(j)})^b(\lambda_{\sigma(r)}\lambda_{\sigma(s)})^c \in E_{abc}, \text{ for } \sigma \in S_{2n}.$$

If the map

$$(\{i,j\},\{k,l\},\{r,s\}) \mapsto (\lambda_i\lambda_j)^a(\lambda_k\lambda_l)^b(\lambda_r\lambda_s)^c \tag{3.6}$$

is injective, then we can deduce that if

$$e_i \wedge e_j \otimes e_k^* \wedge e_l^* \otimes e_r \wedge e_s \in S_{\mathbb{C}}$$

so does

$$e_{\sigma(i)} \wedge e_{\sigma(j)} \otimes e_{\sigma(k)}^* \wedge e_{\sigma(l)}^* \otimes e_{\sigma(r)} \wedge e_{\sigma(s)}, \forall \sigma \in S_{2n}. \tag{3.7}$$

It is not difficult to prove that for an adequate choice of $a, b, c \in \mathbb{Z}$, (3.6) is injective. Hence, since $S_{\mathbb{C}}$ is contained in the $(3,3)$-part of

$$A^{2*}_{1\mathbb{C}} \otimes A^2_{1\mathbb{C}} \otimes A^2_{2\mathbb{C}},$$

then (3.7) has to be of Hodge type $(3,3)$, for any $\sigma \in S_{2n}$.
As $n \geq 4$, it is immediate that we can always find $\sigma \in S_{2n}$ so that an element of the form (3.7) has Hodge type $(4,2)$.
Thus, an element as in (3.5) does not exist in $S_{\mathbb{C}}$, which implies that $S = \{0\}$. □

Remark 3.2.8. The proof shows also that there are not rational Hodge classes in $A^2_{1\mathbb{Q}} \otimes A^2_{1\mathbb{Q}} \cong Hom_{\mathbb{Q}}(A^2_{1\mathbb{Q}}, A^2_{2\mathbb{Q}})$. Hence, there are not morphisms of Hodge structure between these two spaces and they can not be isomorphic.

Another consequence of the transitivity of ϕ_T^* is the simplicity of the Hodge structure of the $A^2_{i\mathbb{Q}}$'s.

Proposition 3.2.9. *The Hodge structures on $A^2_{1\mathbb{Q}}$ and $A^2_{2\mathbb{Q}}$ are simple, i.e. do not contain any non-trivial sub-Hodge structure.*

Proof. By contradiction, suppose that there is a proper non-zero simple sub-Hodge structure

$$K \subset H^2(T, \mathbb{Q}).$$

Since ϕ_T^* is transitive on $H^2(T, \mathbb{Q})$, we have that

$$\exists l > 1 \text{ s.t. } H^2(T, \mathbb{Q}) \cong K^l.$$

But then $H^2(T, \mathbb{Q})$ admits a projector which is an endomorphism of Hodge structure. This contradicts the fact, shown in the proof of Lemma (3.2.11), that the algebra of endomorphisms of Hodge structure of $H^2(T, \mathbb{Q})$ is generated by ϕ_T^* and thus does not contain projectors by property $(*)$.

□

From the proof of Proposition (3.2.9) we can readily deduce the following

Corollary 3.2.10. *Given a morphism of Hodge structure $\tau : A^2_{1\mathbb{Q}} \to H$, where H is a rational Hodge structure, then either τ is injective or τ is the zero morphism.*
Moreover, if τ is injective and there is sub-Hodge structure $H' \subset H$ such that $H' \cap \tau(A^2_{1\mathbb{Q}}) \neq \{0\}$ then $\tau(A^2_{1\mathbb{Q}}) \subset H'$.
The same conclusions are true with $A^2_{1\mathbb{Q}}$ replaced by $A^2_{2\mathbb{Q}}$.

We know that there are morphisms of Hodge structure in $Hom_{\mathbb{Q}}(A^2_{i\mathbb{Q}}, A^2_{i\mathbb{Q}})$, e.g. Id and the one induced by ϕ. We want to know how wide this space is. With the next lemma, we will be able to provide a complete description of the structure of such space.

Lemma 3.2.11. *Up to a coefficient, there are only finitely many elements*

$$\beta \in Hdg(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \otimes \mathbb{C}$$

of rank 1, i.e. of the form $\alpha_1\alpha_2$, $\alpha_i \in A^2_{i\mathbb{Q}} \otimes \mathbb{C}$.

Proof. Following the same notations as in Lemma (3.2.6), we identify Hodge structure on $A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$ to the one on

$$\bigwedge^2 \Gamma^*_{\mathbb{Q}} \otimes \bigwedge^2 \Gamma_{\mathbb{Q}}.$$

Vectors $e_i^* \in \Gamma^*_{\mathbb{C}}$, $i > n$, have Hodge type $(1,0)$, while vectors $e_i \in \Gamma_{\mathbb{C}}$, $i \leq n$, have Hodge type $(1,0)$.
Again, the space $S_{\mathbb{C}} := Hdg(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \otimes \mathbb{C}$ is stable under the action of commuting operators $\wedge^2\phi^t \otimes Id$ and $Id \otimes \wedge^2\phi$, hence it is generated by certain eigenvectors for both of these endomoprhisms:

$$e_i^* \wedge e_j^* \otimes e_k \wedge e_l.$$

Since this space is defined over $\mathbb{Q}$, it has to be stable for the action of S_{2n}. For any permutation $\sigma \in S_{2n}$ of $\{1, \ldots, 2n\}$, if $e_i^* \wedge e_j^* \otimes e_k \wedge e_l \in S_{\mathbb{C}}$, i.e. has Hodge type $(2,2)$, then also

$$e^*_{\sigma(i)} \wedge e^*_{\sigma(j)} \otimes e_{\sigma(k)} \wedge e_{\sigma(l)}$$

has to be of type $(2,2)$. Then, we immediately conclude that this can happen if and only if $i = k$ and $j = l$. Indeed, if the four indices were distinct, by changing them by some $\sigma \in S_{2n}$, we may arrange that $e^*_{\sigma(i)} \wedge e^*_{\sigma(j)} \otimes e_{\sigma(k)} \wedge e_{\sigma(l)}$ has Hodge type $(4,0)$. If e.g. $i = k$, $j \neq l$, by changing by some $\sigma' \in S_{2n}$, we may arrange that $e^*_{\sigma'(i)} \wedge e^*_{\sigma'(j)} \otimes e_{\sigma'(k)} \wedge e_{\sigma'(l)}$ has Hodge type $(3,1)$.
Thus, we have proved that S is generated by the elements $e^*_i \wedge e^*_j \otimes e_i \wedge e_j$, $1 \leq i < j \leq 2n$. It is clear now that S can contain (up to scalar multiplication) only finitely many elements of rank 1 , namely the elements above. □

Remark 3.2.12. We have seen that $S_{\mathbb{C}}$ is generated by elements of the form $e^*_i \wedge e^*_j \otimes e_i \wedge e_j$. These are nothing but the projectors on the eigenspaces of $\phi^*_{\hat{T}}$. Clearly,

$$Id_{A^2_{2\mathbb{C}}} = \sum_{i,j} e^*_i \wedge e^*_j \otimes e_i \wedge e_j \in S_{\mathbb{C}},$$

in fact the identity is also a morphism of Hodge structure, indipendently by the field of definition.
Hence $S_{\mathbb{C}}$ is the algebra generated over $\mathbb{C}$ by Id and $\phi^*_{\hat{T}}$ and S is the algebra generated by the same elements, this time over $\mathbb{Q}$.

3.2.2 Algebraic subsets of $H^*(X', \mathbb{Q})$

Let $D \subset H^2(X', \mathbb{Q})$ be the subspace generated by degree 2 Hodge classes. At the beginning of Section 3.2, we saw that the following decomposition in sub-Hodge structures holds:

$$H^2(X', \mathbb{Q}) = D \oplus A^2. \tag{3.8}$$

Furthermore, since $H^{odd}(T/\pm Id, \mathbb{Q}) = H^{odd}(\hat{T}/\pm Id, \mathbb{Q}) = 0$, by definition

$$A^2 = A^2_1 \oplus A^2_2. \tag{3.9}$$

In view of (3.8), (3.9), any $\alpha \in H^2(X', \mathbb{C})$ can be decomposed in the form

$$\alpha = \alpha_D + \alpha', \ \alpha' = \alpha_1 + \alpha_2. \tag{3.10}$$

We are interested in examining certain algebraic subsets of the cohomology ring of X' that will naturally arise, later on, in this chapter. Recall that, thanks to Deligne's Lemma (2.2.2), we have an easy criterion to recognize sub-Hodge structures of $H^*(X', \mathbb{Q})$ whose generators lie in algebraic subsets of $H^*(X', \mathbb{C})$. This is the main reason for investigating the existence and the form of algebraic subsets.
Let $Z \subset H^2(X', \mathbb{C})$ be the algebraic set of the form

$$Z := \{\alpha \in H^2(X', \mathbb{C}) \mid \alpha^2 = 0 \in H^4(X', \mathbb{C})\}.$$

Z contains the algebraic subsets

$$
\begin{aligned}
Z_1 &= \{\alpha \in H^2(X',\mathbb{C}) \mid \alpha_2 = 0,\ \alpha_1^2 = \alpha_1\alpha_D = \alpha_D^2 = 0 \in H^4(X',\mathbb{C})\} \\
Z_2 &= \{\alpha \in H^2(X',\mathbb{C}) \mid \alpha_1 = 0,\ \alpha_2^2 = \alpha_2\alpha_D = \alpha_D^2 = 0 \in H^4(X',\mathbb{C})\}
\end{aligned}
$$

The following proposition shows what are the irreducible component of the Z_i in A^2 and will play a fundamental role in the prosecution of the chapter

Proposition 3.2.13. *Any irreducible component of Z_1 (resp. Z_2) containing*

$$Z_{1,0} := Z_1 \cap \{\alpha = \alpha' + \alpha_D \mid \alpha_D = 0\} \text{ (resp. } Z_{2,0} := Z_2 \cap \{\alpha = \alpha' + \alpha_D \mid \alpha_D = 0\})$$

is an irreducible component if Z. The decomposition $\alpha = \alpha' + \alpha_D$ is the same as in (3.10)

Proof. Given $\alpha \in H^2(X',\mathbb{C}),\ \alpha \in Z$ if and only if $\alpha^2 = 0 \in H^4(X',\mathbb{C})$ which is equivalent to

$$\alpha'^2 + 2\alpha_D\alpha' + \alpha_D^2 = 0. \tag{3.11}$$

Since all the classes in D are of Hodge type $(1,1)$, $\alpha_D\alpha'$ belongs to $N_2H^4(X')\otimes\mathbb{C}$, where $N_2H^4(X')$ is the maximal sub-Hodge structure of $H^4(X',\mathbb{Q})$ of level 2.
As a conqequence of equation (3.11), we claim that α'^2 belongs to $N_2A^4_{\mathbb{Q}}$, where again N_2 indicates the maximal sub-Hodge structure of level 2: indeed, $\alpha'^2 = -(2\alpha_D\alpha' + \alpha_D^2)$ and $\alpha_D^2 \in H^{2,2}(X') \subset N_2A^4_{\mathbb{Q}}$, while

$$2\alpha_D\alpha' = 2\sum_{i=1,2}\alpha_D\alpha_i = 2\sum_{i=1,2}\ \sum_{(p,q)=(2,0),(1,1),(0,2)}\alpha_D\alpha_i^{(p,q)}.$$

Here

$$\sum_{(p,q)=(2,0),(1,1),(0,2)}\alpha_i^{(p,q)},\ i = 1,2$$

is the Hodge decomposition of α_i in $H^2(X',\mathbb{C})$. But if $\alpha_i^{(p,q)} \in H^{(p,q)}(X')$ then $\alpha_D\alpha_i^{(p,q)} \in H^{(p+1,q+1)}(X')$. Hence $2\alpha_D\alpha' \in N_2H^4(X')$, which proves the claim.
By the Künneth formula, we can decompose $A^4_{\mathbb{Q}}$ into the direct sum of sub-Hodge structures of weight 4

$$A^4_{\mathbb{Q}} = A^4_{1\mathbb{Q}} \oplus (A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \oplus A^4_{2\mathbb{Q}}. \tag{3.12}$$

Now, remember that

$$\alpha'^2 = \alpha_1^2 + 2\alpha_1\alpha_2 + \alpha_2^2 \in N_2A^4_{\mathbb{Q}} \otimes \mathbb{C}. \tag{3.13}$$

From the decomposition (3.12) into sub-Hodge structures, it follows that any term of (3.13) has to be in the maximal level 2 structure of the corresponding summand. Since by Lemma (3.2.5), $A^4_{i\mathbb{Q}},\ i = 1,2$ have no level 2 sub-Hodge structure,

$$\alpha_1^2 = 0,\ \alpha_2^2 = 0.$$

In view of this, we can write (3.11) in the new form

$$2\alpha_1\alpha_2 + 2\alpha_D\alpha' + \alpha_D^2 = 0. \tag{3.14}$$

As already noticed, $\alpha_1\alpha_2$ belongs to $N_2(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}})$. Actually, we can say more: considering

$$D \cdot A^2_{\mathbb{Q}} + D^2 \subset H^4(X', \mathbb{Q}),$$

it can be thought as given by the following surjective map

$$A^2_{1\mathbb{Q}} \otimes D \oplus D \otimes D \oplus A^2_{2\mathbb{Q}} \otimes D \to D \cdot A^2_{\mathbb{Q}} + D^2 \tag{3.15}$$

which is given on a basis by

$$(\alpha_1 \otimes \beta_1, \gamma \otimes \delta, \alpha_2 \otimes \beta_2) \mapsto \alpha_1\beta_1 + \alpha_2\beta_2 + \gamma\delta\ .$$

Thus, $D \cdot A^2_{\mathbb{Q}} + D^2$ is a quotient of direct sum of level 2 Hodge structures isomorphic to $A^2_{i\mathbb{Q}}$, $i = 1, 2$ and a trivial Hodge structure of weight 4.
Summarizing, we have that $\alpha_1\alpha_2$ has to belong to the space

$$N'_2(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \otimes \mathbb{C},$$

where now N'_2 means the maximal sub-Hodge structure of level 2 which is a subquotient of a sum of copies of $A^2_{1\mathbb{Q}}$ or $A^2_{2\mathbb{Q}}$ or a trivial Hodge structure[2].
By the results of the previous subsection, we see that $N'_2(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}})$ is actually equal to the maximal sub-Hodge structure of level 2 of $A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$ which is a sum of copies of $A^2_{1\mathbb{Q}}$ or $A^2_{2\mathbb{Q}}$ or a trivial Hodge structure.
But now by Lemma (3.2.6), we deduce that finally $\alpha_1\alpha_2$ belongs to the maximal trivial sub-Hodge structure of $A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$, that is

$$Hdg(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \otimes \mathbb{C},$$

the space of rational Hodge classes of $A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$. Lemma (3.2.11) tells us that (up to scalar multiplication) there are only finitely many elements of the form $\alpha_1\alpha_2$ in $Hdg(A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}) \otimes \mathbb{C}$.
From the above analysis, we have two different possibilities for $\alpha \in Z$:

- $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and then $\alpha_1\alpha_2$ has to be proportional to one of the finitely many elements of Lemma (3.2.11);
- at least one between α_1 and α_2 has to be 0.

In this last case, we claim that α belongs either to Z_2 or to Z_1, respectively. Indeed, we know that

$$\alpha_1^2 = 0 = \alpha_2^2.$$

[2]Given a vector space V, a subquotient of V is a vector space W such that there exist two non-empty subspaces $R \subset S \subset V$ and an isomorphism $W \cong S/R$.

Assume $\alpha_2 = 0$. Then, equation (3.14) becomes:

$$2\alpha_D\alpha_1 + \alpha_D^2 = 0.$$

For Hodge type reasons, this implies that

$$\alpha_D\alpha_1 = 0 = \alpha_D^2.$$

in fact, α_D^2 belongs to $Hdg^4(X')\mathbb{C}$, while $\alpha_D\alpha_1$ belongs to the space

$$N_2''H^4(X', \mathbb{Q}) \otimes \mathbb{C},$$

defined as the maximal sub-Hodge structure of $H^4(X', \mathbb{Q})$ of level 2 isomoprhic to a subquotient of some power of $A^2_{1\mathbb{Q}}$. But again, by an easy simplicity argument (see proposition (3.2.9)), $N_2''H^4(X', \mathbb{Q})$ has to be the maximal sub-Hodge structure of $H^4(X', \mathbb{Q})$ isomorphic to some power of $A^2_{1\mathbb{Q}}$. But the intersection

$$Hdg^4(X') \cap N_2''H^4(X', \mathbb{Q})$$

has to be zero, since there is no non-zero Hodge class in $A^2_{1\mathbb{Q}}$. Thus, also

$$Hdg^4(X') \otimes \mathbb{C} \cap N_2''H^4(X', \mathbb{Q}) \otimes \mathbb{C}$$

has to be 0. Hence we proved that $2\alpha_D\alpha_1 + \alpha_D^2 = 0$ implies that $\alpha_D\alpha_1 = 0 = \alpha_D^2$.
In conclusion we proved that Z is the set theoretic union of Z_1, Z_2 and of a set which projects to a finite set of lines in $A^2_{1\mathbb{C}}$ and $A^2_{2\mathbb{C}}$.
Let now Z_1' be an irreducible component of Z_1 which contains $Z_{1,0}$. Suppose Z_1' is not an irreducible component of Z. Thus, there should be an irreducible component Z' of Z, not in Z_1, which contains Z_1'. But now, $Z' \setminus Z_1'$ would be dense in Z'. Hence, the projection on $A^2_{1\mathbb{C}}$ of Z_1' would contain $Z_{1,0}$ (which at least contain more than one line), while the projection of $Z_1' \setminus (Z_1' \cap Z_1)$ which is open and irreducible in Z_1' should be contained in the projection of Z_2 (which is 0) and a finite set of lines. But this is impossible for a straightforward connectedness argument. □

We introduce the last two technical lemmas.

Lemma 3.2.14. *Let*

$$D_1 := \{\alpha_D \in D \mid \alpha_D\alpha_1 = 0 \in H^4(X', \mathbb{Q}), \forall \alpha_1 \in A^2_{1\mathbb{Q}}\} \subset D.$$

If $\alpha_D \in D \otimes \mathbb{C}$ satisfies $\alpha_D\alpha_1 = 0$ for some $0 \neq \alpha_1 \in A^2_{1\mathbb{C}}$, then $\alpha_D \in D_1 \otimes \mathbb{C}$.

Proof. First of all, we make a reduction: suppose there is a compact Kähler manifold X'' and a map $\psi' : X'' \to X'$ which is surjective, holomoprhic and of finite degree, and the result of the lemma is true for X'', with D replaced by $Hdg^2(X'')$ and $A^2_{1\mathbb{Q}}$ replaced by $\psi'^*(A^2_{1\mathbb{Q}})$. Then, we claim that the result of the lemma holds for X', too. In fact, since the map has finite degree and X'' is Kähler ψ^* is injective and sends D in $Hdg^2(X'')$.

Recall now, that X' is bimeromorphic to a quotient of the $\mathbb{P}^1 \times \mathbb{P}^1$-bundle $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ over $T \times \hat{T}$. Hence, we have a dominant meromoprhic map from $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ to X'.
Using Hironaka's desingularization theorem and the observation above, we can reduce to the case where X' is obtained from $W := \mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ by a sequence of blow-ups. We first prove that the result is true for W.
The cohomology of degree 2 of

$$W \xrightarrow{\pi} T \times \hat{T}$$

is a free module, generated by $H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$, over the cohomology of $T \times \hat{T}$. Thus the space of degree 2 Hodge classes D on W is the sum of two spaces: D_0, which has rank 2 and is isomorphic to $H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Q})$, and D_1, which is isomorphic via π^* to the set of degree 2 Hodge classes in $H^2(T \times \hat{T}, \mathbb{Q})$.
Recall that

$$H^2(T \times \hat{T}, \mathbb{Q}) = H^2(K \times \hat{K}, \mathbb{Q}) \oplus H^1(T, \mathbb{Q}) \otimes H^1(\hat{T}, \mathbb{Q}).$$

We conclude that D_1 is contained in $H^1(T, \mathbb{Q}) \otimes H^1(\hat{T}, \mathbb{Q})$. Hence, by the proof of Lemma (3.2.11), D_1 is generated over $\mathbb{Q}$ by $p := c_1(\mathcal{P})$ and its pull-backs under $(\phi_T^l)^* \otimes Id,\ l \in \mathbb{N}$. This implies that the cup-product map

$$\begin{array}{rcl} D_1 \otimes \pi^*(H^2(T, \mathbb{Q})) & \to & H^4(W, \mathbb{Q}) \\ (d_1, \alpha) & \mapsto & d_1 \cup \alpha \end{array}$$

is injective. A straightforward calculation shows that also the cup-product map

$$\begin{array}{rcl} D_0 \otimes \pi^*(H^2(T, \mathbb{Q})) & \to & H^4(W, \mathbb{Q}) \\ (d_0, \alpha) & \mapsto & d_0 \cup \alpha \end{array}$$

is injective. Thus, the proof for W is concluded.
We now want to prove the result by induction on the number i of blow-ups of W along smooth centers.
Assume the result of the lemma holds for W_i and let $\tau_{i+1} : W_{i+1} \to W_i$ be the blow-up of a smooth irreducible submanifold $Z \subset W_i$. The set of degree 2 Hodge classes D_{i+1} on W_{i+1} is generated by $\tau_{i+1}^*(D_i)$ and the class of $[\Delta_Z]$ of the exceptional divisor E_Z over Z. Theorem (1.4.8) shows that if there is an equality

$$[\Delta_Z] \cup \tau_{i+1}^*(\alpha) = 0,\ \textit{modulo}\ \tau_{i+1}^*(H^*(W_i, \mathbb{C}))$$

then in fact, $[\Delta_Z] \cup \tau^*(\alpha) = 0 \in H^*(W_{i+1}, \mathbb{C})$.
Suppose there is a relation $\alpha_D \alpha = 0 \in H^*(W_{i+1}, \mathbb{C}),\ \alpha_D \in D_{i+1} \otimes \mathbb{C}$ and $\alpha \in \pi^*(H^2(T \times \{0\}, \mathbb{C}))$. Writing

$$\alpha_D = \mu[\Delta_Z] + \alpha'_D,\ \mu \in \mathbb{C},\ \alpha'_D \in \tau_{i+1}^*(D_i) \otimes \mathbb{C},$$

we conclude, as above, that $\mu[\Delta_Z]\cup\alpha = 0 \in H^*(W_{i+1}, \mathbb{C})$. This means that either $\mu = 0$, hence we can apply the inductive step to α'_D, or

$$[\Delta_Z] \cup \alpha = 0 \in H^*(W_{i+1}, \mathbb{C}).$$

Since multiplication by the Hodge class $[\Delta_Z]$ is a morphism of Hodge structure from $H^2(T, \mathbb{Q})$ to $H^4(W_{i+1}, \mathbb{Q})$, its kernel is a sub-Hodge structure of $H^2(T, \mathbb{Q})$. So this map is either injective or 0, as noted in Corollary (3.2.10).
Finally, if there is one non-zero α satisfying $\alpha_D\alpha = 0 \in H^*(W_{i+1}, \mathbb{C})$ and the coefficient μ is non-zero, then $[\Delta_Z] \cup \alpha' = 0 \in H^*(W_{i+1}, \mathbb{C})$, $\forall \alpha' \in \pi^*(H^2(T \times \{0\}, \mathbb{Q}))$. Then, the equality $\alpha_D\alpha = 0$ reduces to $\alpha'_D\alpha = 0$, which already holds in $H^*(W_i, \mathbb{C})$. □

Lemma 3.2.15. *1. For any $d \in D \otimes \mathbb{C}$ and any $\beta \in A^{4n-2}_{\mathbb{C}} \subset H^{4n-2}(X', \mathbb{C})$ we have*

$$d^3\beta = 0 \in H^{4n+4}(X', \mathbb{C}).$$

2. The complex subspace $D \otimes \mathbb{C} \subset H^2(X', \mathbb{C})$ is an irreducible component of the algebraic set

$$Z' := \{d \in H^2(X', \mathbb{C}) \mid d^3\beta = 0, \forall \beta \in A^{4n-2}_{\mathbb{C}}\}.$$

Proof. 1. D has a trivial Hodge structure. Thus, for any $d \in D$ the map

$$\alpha \mapsto d^3\alpha \in H^{4n+4}(X', \mathbb{Q}) \cong \mathbb{Q}$$

is a morphism of Hodge structure, hence it can be identified, for Poincaré duality to a Hodge class in $(A^{4n-2}_{\mathbb{Q}})^* = A^2_{\mathbb{Q}}$. We already know that $A^2_{\mathbb{Q}}$ has no non-zero Hodge classes and the proof ends.

2. See [V05, §2, Lemma 6].

□

3.2.3 The regular locus of ψ

We conclude this section with a Proposition concerning the geometry of the bimeromorphic map

$$\psi : X' \dashrightarrow X.$$

This will be essential in the sequel.
At the beginning of this section, we proved that the map

$$q \circ \psi : X' \dashrightarrow (T/\pm Id) \times (\hat{T}/\pm Id)$$

is holomorphic.
Let us denote $X'_0 := (q \circ \psi)^{-1}(K_0 \times \hat{K}_0)$.

Proposition 3.2.16. *There exists a dense Zariski open set $U \subset K_0 \times \hat{K}_0$ such that, if we denote*

$$X'_U := (q \circ \psi)^{-1}(U),\ X_U := q^{-1}(U),$$

the induced meromorphic map

$$\psi : X'_U \dashrightarrow X_U$$

is holomorphic.

Lemma 3.2.17. *The only closed irreducible positive dimensional proper analytic subsets of $T \times \hat{T}$ are of the form $\{x\} \times \hat{T},\ x \in T$ or $T \times \{y\}, y \in \hat{T}$.*

Proof. Firts of all, recall that T and $\hat{T}$ do not contain positive dimensional proper analytic subsets as noted in Remark (2.1.4).
It follows that if $Z \subset T \times \hat{T}$ is positive dimensional proper and irreducible, not of the above form, then it must be étale over both T and $\hat{T}$. Indeed, this is a consequence of what we have just noted and of the following

Theorem 3.2.18. *Let B a proper irreducible analytic subset of a complex torus A. Then there exist a complex subtorus A_1 of A and a projective variety L, which is a subvariety of an abelian variety, such that there is an analytic surjective morphism $\mu : B \to L$ whose generic fibre is A_1.*

Proof. [U75, Ch. 4, Thm. 10.9] □

But if Z were étale over T and $\hat{T}$, its cohomology class $[Z] \in H^{2n}(T \times \hat{T}, \mathbb{Q}) \cong \bigoplus_{p+q=2n} H^p(T,\mathbb{Q}) \otimes H^q(\hat{T},\mathbb{Q})$ (more precisely its projection on $H^{2n-1}(T,\mathbb{Q}) \otimes H^1(\hat{T},\mathbb{Q})$) would give an isomorphism between $H^1(T,\mathbb{Q})$ and $H^1(\hat{T},\mathbb{Q})$. We have already seen that this is not the possible (see Remark 3.2.8). □

Lemma 3.2.19. *The only irreducible proper closed analytic subsets of $\mathbf{P}(E)$ (resp. $\mathbf{P}(E_\phi)$) which dominate $T \times \hat{T}$ are the images Σ_1, Σ_2 (resp. $\Sigma_1^\phi, \Sigma_2^\phi$) of the two natural sections σ_1, σ_2 (resp. $\sigma_1^\phi, \sigma_2^\phi$) of $\mathbf{P}(E)$ (resp. $\mathbf{P}(E_\phi)$) corresponding to the splitting*

$$E = \mathcal{P} \oplus \mathcal{P}^{-1} \text{ (resp. } E_\phi = \mathcal{P}_\phi \oplus \mathcal{P}_\phi^{-1}).$$

Proof. Let $Z \subset \mathbb{P}(E)$ be an hypersurface dominating $T \times \hat{T}$. We will denote by $e : Z \to T \times \hat{T}$ the generically finite map. By the previous lemma, Z has to contain a dense Zariski open set Z_0 which is an étale cover of a Zariski open set $U \subset T \times \hat{T}$.On the contrary, $U_1 := T \times \hat{T} \setminus U$ is an union of analytic proper subsets of the form

$$\{x\} \times \hat{T},\ x \in T, \text{ or } T \times \{y\},\ y \in \hat{T}.$$

Let us note that Z induces a section of the induced $\mathbb{P}^1$-bundle

$$\mathbb{P}(E)_Z := e^*\mathbb{P}(E).$$

Such a section is given by a line bundle over Z, $\mathcal{L}$, and a surjective map

$$E^* := e^*\mathcal{P} \oplus e^*\mathcal{P}^{-1} \to \mathcal{L}. \quad (3.16)$$

Indeed, a section of $\mathbb{P}(E)_Z$ is given by a 1-dimensional subspace of E^* varying regularly. Hence, the map in (3.16) is simply given by the quotient of E^* for that subspace.
If one of the two induced maps

$$e^*\mathcal{P} \to \mathcal{L}, \text{ or } e^*\mathcal{P}^{-1} \to \mathcal{L}$$

is zero, then Z is contained in Σ_1 or Σ_2 (actually it has to be equal). Otherwise, both $e^*\mathcal{P}^{-1} \otimes \mathcal{L} = Hom(e^*\mathcal{P}, \mathcal{L})$ and $e^*\mathcal{P} \otimes \mathcal{L} = Hom(e^*\mathcal{P}^{-1}, \mathcal{L})$ have non-zero sections. As the codimension of U_1 is ≥ 2, we find that there is some $k \in \mathbb{N}$ such that $\mathcal{L}^{\otimes k}$ is isomorphic to $e^*(\mathcal{K})$ on Z_0, for some line bundle $\mathcal{K}$ on U, hence also on $T \times \hat{T}$.
Thus,

$$e^*\mathcal{P}^{-k} \otimes \mathcal{L}^{\otimes k} = e^*(\mathcal{P}^{-k} \otimes \mathcal{K})$$

and

$$e^*\mathcal{P}^{k} \otimes \mathcal{L}^{\otimes k} = e^*(\mathcal{P}^{k} \otimes \mathcal{K})$$

have non-zero sections on Z_0. It follows that for some $m \in \mathbb{N}$, there are non-zero sections of

$$\mathcal{P}^{-km} \otimes \mathcal{K}^{\otimes m} \text{ and } \mathcal{P}^{km} \otimes \mathcal{K}^{\otimes m}$$

on U, hence also on $T \times \hat{T}$.
Since $T \times \hat{T}$ does not contain hypersurfaces, these sections do not vanish anywhere. It is a consequence that $\mathcal{P}^{-km} \cong \mathcal{P}^{km}$, which is not possible since their Chern classes are different.
Hence the lemma is proved for $\mathbb{P}(E)$ and the result for $\mathbb{P}(E_\phi)$ follows, as

$$\mathbb{P}(E) \cong \mathbb{P}((\phi^{-1}, Id)^* E_\phi).$$

□

Corollary 3.2.20. *1. The only irreducible codimension* 1 *analytic subsets of*

$$\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$$

dominating $T \times \hat{T}$ *are of the form* $pr_1^{-1}(\Sigma_i)$, $i = 1, 2$ *or* $pr_2^{-1}(\Sigma_i^\phi)$, $i = 1, 2$.

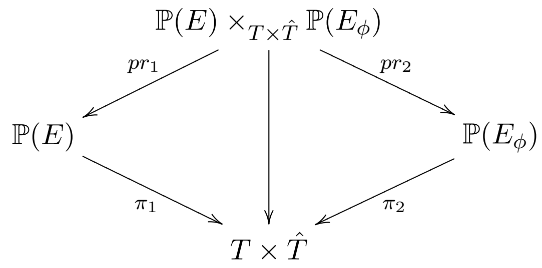
2. The only irreducible codimension 2 *analytic subsets of*

$$\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$$

dominating $T \times \hat{T}$ *are complete intersections*

$$pr_1^{-1}(\Sigma_i) \cap pr_2^{-1}(\Sigma_j^\phi),\ i = 1, 2,\ j = 1, 2.$$

Proof. Recall that by definition of fibre product we have a commutative diagram of the form



1. Let Z be an irreducible hypersurface of $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$. Let us consider $pr_1(Z)$, then there are two possibilities:

   - $pr_1(Z) = \mathbb{P}(E)$;
   - $pr_1(Z)$ dominates $T \times \hat{T}$ and has codimension 1 in $\mathbb{P}(E)$, hence it is exactly one of the $\Sigma_i,\ i = 1, 2$ of the previous lemma.

   If $pr_1(Z) = \Sigma_i$, then $Z = pr_1^{-1}(\Sigma_i)$.
   If $pr_1(Z) = \mathbb{P}(E)$, let us consider on $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ the line bundle $\mathcal{L} := \mathcal{O}(Z)$ and let $H = pr_1^*(\mathcal{O}_{\mathbb{P}(E)}(1))$. By the Leray-Hirsch Theorem, since $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $T \times \hat{T}$,

   $$\mathcal{L} = H^{\otimes\alpha} \otimes pr_2^*(\mathcal{K}),$$

   for some line bundle $\mathcal{K}$ on $\mathbb{P}(E_\phi)$. Thus, we have that

   $$R^0 pr_{2*}\mathcal{L} = Sym^\alpha(\pi_2^* E) \otimes \mathcal{K} = Sym^\alpha(\pi_2^*\mathcal{P} \oplus \pi_2^*\mathcal{P}^{-1}) \otimes \mathcal{K}.$$

   Here α will be non-negative, as $\mathcal{L}$ has non-zero sections.
   The non-zero section of $\mathcal{L}$ defining Z gives rise to sections $\sigma_{\gamma\gamma'}$ of $\pi_2^*\mathcal{P}^{\otimes\gamma} \otimes \pi_2^*\mathcal{P}^{\gamma'} \otimes \mathcal{K}$, for $\gamma, \gamma' \geq 0,\ \gamma + \gamma' = \alpha$.
   Note that only one $\sigma_{\gamma\gamma'}$ can be non-zero. Indeed, by Lemma (3.2.19), the divisors of $\sigma_{\gamma\gamma'}$ are combinations of Σ_1^ϕ and Σ_2^ϕ and the two line bundles differ by a multiple of $\pi_2^*\mathcal{P}_\phi$. Thus, if there were non-zero sections $\sigma_{\gamma\gamma'}$ for at least two distinct couples $(\gamma = a, \gamma' = b),\ (\gamma = c, \gamma' = d),\ a + b = \alpha = c + d$, then we would get a proportionality relation between $\pi_2^*\mathcal{P}_\phi$ and $\pi_2^*\mathcal{P}$ on $\mathbb{P}(E_\phi)$, which is clearly impossible. Thus there is only one non-zero section.
   Let $\sigma_{\gamma\gamma'}$ be such section. There are now two possibilities:

   - the divisor $D_{\gamma\gamma'}$ of $\sigma_{\gamma\gamma'}$ is non-empty;
   - the divisor $D_{\gamma\gamma'}$ of $\sigma_{\gamma\gamma'}$ is empty.

   In the first case, as Z is irreducible and contains $pr_2^{-1}(D_{\gamma\gamma'})$, then Z must be the pull-back of a divisor on $\mathbb{P}(E_\phi)$ and Lemma (3.2.19) gives the result.

Otherwise, if the divisor $D_{\gamma\gamma'}$ of $\sigma_{\gamma\gamma'}$ is empty, one concludes that the line bundle $\mathcal{K}$ is a pull-back

$$\mathcal{K} = \pi_2^* \mathcal{K}',$$

for some line bundle $\mathcal{K}'$ on $T \times \hat{T}$. In this case, $\mathcal{L}$ is also a pull-back

$$\mathcal{L} = pr_1^* \mathcal{L}',$$

for some line bundle $\mathcal{L}'$ on $\mathbb{P}(E)$ and thus Z is equal to $pr_1^{-1}(Z')$, for some $Z' \subset \mathbb{P}(E)$. Again, Lemma (3.2.19) gives the result.

2. The proof is obtained by projecting codimension 2 subsets of $\mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi)$ to $\mathbb{P}(E)$ and $\mathbb{P}(E_\phi)$.

□

The results above allow us to describe the codimension 1 and codimension 2 proper analytic subsets of $Q := \mathbb{P}(E) \times_{T\times\hat{T}} \mathbb{P}(E_\phi) / < (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$, which dominate $K \times \hat{K}$. These are the images in Q of the subvarieties described above.
Let us note that the two hypersurfaces $pr_1^{-1}(\Sigma_1)$, $pr_1^{-1}(\Sigma_2)$ descend to only one irreducible hypersurface

$$\Sigma \subset Q.$$

In fact, the two factors of the splitting $E = \mathcal{P} \oplus \mathcal{P}^{-1}$ are exchanged under i and $\hat{i}$, so $pr_1^{-1}(\Sigma_1)$, $pr_1^{-1}(\Sigma_2)$ are permuted by $< (i, i_\phi), (\hat{i}, \hat{i}_\phi) >$. For the same reason $pr_1^{-1}(\Sigma_1^\phi)$, $pr_1^{-1}(\Sigma_2^\phi)$ give rise to only one hypersurface

$$\Sigma_\phi \subset Q.$$

Thus, Q contains only two hypersurfaces

$$\Sigma, \Sigma_\phi \tag{3.17}$$

and one codimension 2 analytic subvariety

$$W = \Sigma \cap \Sigma_\phi, \tag{3.18}$$

dominating $K \times \hat{K}$.
Sketch of the proof of Proposition (3.2.16). The proof is now immediate from the analysis above. Starting from X, the only modification, which we can do, whose center dominates $K \times \hat{K}$, is to blow-up W. In the blown-up variety, we have as divisors the exceptional divisors, the proper transforms of the divisors Σ, Σ_ϕ and they are the only one. Furthermore, the only codimension 2 closed analytic subset dominating $K \times \hat{K}$ is the union of two copies of W, indexed by the choice of one of the divisors Σ, Σ_ϕ, since $W = \Sigma \cap \Sigma_\phi$. The same situation happens each time we blow-up one copy of W appearing in the previous step.
The key point is now the following:

If the map $\psi : X' \dashrightarrow X$ is not defined over the generic point of $K \times \hat{K}$, we can see it as a birational map between surface bundles over the generic point of $K \times \hat{K}$. Then after a finite sequence of blow-ups of X along codimension 2 subsets dominating $K \times \hat{K}$, some divisor $D \subset X$ in the blown-up variety must be generically contractible over $K \times \hat{K}$, i.e. be made of a disjoint union of rational curves of self-intersection -1 in the generic surface X_t. D has to project to a divisor in X. This follows from the factorization of birational map between surfaces (see [Bea96]). As this divisor dominates $K \times \hat{K}$, it must be one of those described above, that is a proper transform of Σ, Σ_ϕ.
The contradiction comes from the fact that after the blow-up of W, the proper transforms of Σ and Σ_ϕ are families of rational curves of self-intersection -2, and the value of the self-intersection can only decrease after further blow-ups. One the other hand, if we do not blow-up anything, these divisors are families of curves of self-intersection 0, which do not contract.

■

3.3 Proof of Theorem (3.1.2)

In this section, we assume the hypotheses of Theorem (3.1.2), namely, X' is bimeromorphically equivalent to X and Y is a compact Kähler manifold such that there is an isomorphism

$$\gamma : H^*(Y) \xrightarrow{\cong} H^*(X', \mathbb{Q})$$

of cohomology algebras. Our goal is proving that Y is not a projective manifold.
The argumentation will be based on the analysis of the algebra $H^*(X', \mathbb{Q})$ realized in the previous section and on Deligne's Lemma (2.2.2).
Notations are as in the previous sections.
Our first step is the following:

Proposition 3.3.1. *Let X', Y, γ be as in Theorem (3.1.2). Then $\gamma^{-1}(A^2_{i\mathbb{Q}})$, $i = 1, 2$ are rational sub-Hodge structures of $H^2(Y, \mathbb{Q})$.*

Proof. We just have to show how to recover the space $A^2_{1\mathbb{C}} = A^2_{1\mathbb{Q}} \otimes \mathbb{C}$ (resp. $A^2_{2\mathbb{C}} = A^2_{2\mathbb{Q}} \otimes \mathbb{C}$) as generated by an algebraic subset of $H^2(X', \mathbb{C})$ defined using the ring structure of the cohomology ring of Y. Via γ, we shall recover analogously $\gamma^{-1}(A^2_{1\mathbb{C}})$ (resp. $\gamma^{-1}(A^2_{1\mathbb{C}})) \subset H^2(Y, \mathbb{C})$ through the image of the algebraic subset previously determined. At that point, the proof will be concluded by means of Deligne's Lemma.
We give the proof only for $\gamma^{-1}(A^2_{1\mathbb{Q}})$, the other case being identical.
By Proposition (3.2.13), the irreducible components of the algebraic subset

$$Z_1 = \{\alpha_1 + d,\ d \in D_{\mathbb{C}} \mid \alpha_1 \in A^2_{1\mathbb{C}},\ \alpha_1^2 = 0, d^2 = 0, \alpha_1 d = 0\}$$

which contains the algebraic subsets

$$Z_{1,0} := \{\alpha \in A^2_{1\mathbb{C}} \mid \alpha^2 = 0\}$$

are irreducible components of

$$Z = \{\alpha \in H^2(X', \mathbb{C}) \mid \alpha^2 = 0\}.$$

Now, Lemma (3.2.14) implies that if we denote by D_1 the $\mathbb{Q}$-vector subspace of $H^2(X', \mathbb{Q})$ defined as $D_1 := \{d \in D \mid d\alpha = 0, \forall \alpha \in A^2_{1\mathbb{Q}}\}$, the condition

$$\alpha_1 d = 0 \in H^4(X', \mathbb{C}),\ d \in D \otimes \mathbb{C}$$

for some non-zero

$$\alpha_1 \in A^2_{1\mathbb{C}},$$

implies that $d \in D_{1\mathbb{C}} := D_1 \otimes \mathbb{C}$.
As a consequence of this Lemma, considering now the algebraic subset of $H^2(X', \mathbb{C})$,

$$Z_1' := \{\alpha_1 + d,\ d \in D_1 \otimes \mathbb{C} \mid \alpha_1 \in A^2_{1\mathbb{C}},\ \alpha_1^2 = 0, d^2 = 0\}, \tag{3.19}$$

it is clear that the irreducible components of Z'_1 containing $Z_{1,0}$ are irreducible components of Z. Since $A^2_{1\mathbb{C}}$ is defined over $\mathbb{Q}$ and it is generated by its algebraic subset $Z_{1,0}$, it remains only to show how to recover $Z_{1,0}$ from Z'_1.
Let

$$D'_{1\mathbb{C}} \subset D_{1\mathbb{C}}$$

be the complex vector space generated by the algebraic subset

$$Z_{D1} := \{d \in D_{1\mathbb{C}} \mid d^2 = 0\}.$$

It is clear that $D'_{1\mathbb{C}}$ is defined over $\mathbb{Q}$, i.e.

$$D'_{1\mathbb{C}} = D'_1 \otimes \mathbb{C}$$

for some rational subspace $D'_1 \subset H^2(X', \mathbb{Q})$, since $D_{1\mathbb{C}}$ itself is generated over $\mathbb{Q}$ and the equations cutting out Z_{D1} have rational coefficients.
If $D'_1 = 0$, thus $Z'_1 = Z_{1,0}$. In general, (3.19) shows that Z'_1 is the product of Z_{D1} and $Z_{1,0}$ in $D'_1 \oplus A^2_1\mathbb{C}$.
Now, there are two possibilities:

1. $Z_{D1} \neq D'_{1\mathbb{C}}$;

2. $Z_{D1} = D'_{1\mathbb{C}}$.

In 1. we can easily recover $Z_{1,0}$ as a subvariety of the singular locus of the product $Z_{D1} \times Z_{1,0} \subset D'_1 \oplus A^2_{\mathbb{C}}$. Indeed, in this case, $0 \in Z_{D1}$ is a singular point, hence $Z_{1,0} \times \{0\}$ is a subvariety of the singular locus of the product (note that we have used exclusively the algebra structure of $H^*(X', \mathbb{C})$).
We have only to exclude the possibility that

$$D'_1 \neq 0,\ Z_{D1} = D'_{1\mathbb{C}}. \tag{3.20}$$

Assume (3.20) holds. As D'_1 is a $\mathbb{Q}$-vector space, there would be a non-zero real element $d \in D \subset H^{1,1}_{\mathbb{R}}(X')$ such that

$$d^2 = 0,\ d\alpha = 0, \forall \alpha \in A^2_{1\mathbb{R}}.$$

But there exists also a non-zero

$$\alpha \in A^{1,1}_{\mathbb{R}} := H^{1,1}_{\mathbb{R}}(X') \cap A^2_{1\mathbb{R}}$$

such that $\alpha^2 = 0$. Then, the rank 2 real vector space

$$B :=< d, \alpha >\subset H^{1,1}_{\mathbb{R}}(X')$$

satisfies the property

$$\forall \theta \in B,\ \theta^2 = 0.$$

But this contradicts the Hodge index theorem. Indeed, X' is dominated by a Kähler compact manifold (since it is bimeromorphic to X), i.e. there is a Kähler compact manifold K and a holomorphic 1-1 surjective map

$$\tau : K \to X$$

such that pull-back maps $\tau^* : H^*(X, \mathbb{R}) \to H^*(K, \mathbb{R})$ are injective and preserve the strong Hodge decomposition. Choosen a Kähler class $\omega \in H^{1,1}_{\mathbb{R}}(K)$, this gives an intersection form on $H^{1,1}_{\mathbb{R}}(K)$, $q_\omega(c) = \int_K \omega^{2n} \wedge c \wedge c$, $\forall c \in H^{1,1}_{\mathbb{R}}(K)$ that admits only one positive sign. Hence the contradiction is realized by the existence of the rank 2 vector subspace of $H^{1,1}_{\mathbb{R}}(K)$ $\tau^*(B)$, which is isotropic for the intersection q_ω. □

Corollary 3.3.2. *With the same assunmptions and notations, the subspace*

$$\gamma^{-1}(D) \subset H^2(Y, \mathbb{Q})$$

is a rational sub-Hodge structure.

Proof. By Lemma (3.2.15), 2., $D \otimes \mathbb{C}$ is an irreducible component of the set

$$Z'' = \{d \in H^2(X', \mathbb{C}) \mid d^3\beta = 0,\ \forall \beta \in A^{4n-2}_{\mathbb{C}}\}.$$

It follows that $\gamma^{-1}(D) \otimes \mathbb{C}$ is an irreducible component of the set

$$\gamma^{-1}(Z'') = \{d \in H^2(Y, \mathbb{C}) \mid d^3\beta = 0,\ \forall \beta \in \gamma^{-1}(A^{4n-2}_{\mathbb{C}})\}.$$

By the previous proposition $\gamma^{-1}(A^{4n-2})$ is a rational sub-Hodge structure of H^{4n-2}, being the degree $4n-2$ piece of the subalgebra generated by $\gamma^{-1}(A^2)$.
Hence its annihilator

$$\gamma^{-1}(A^{4n-2})_0 = \{\delta \in H^6(Y, \mathbb{Q}) \mid \delta\beta = 0,\ \forall \beta \in \gamma^{-1}(A^{4n-2})_{\mathbb{C}}\}$$

is also a rational sub-Hodge structure of $H^{4n-2}(Y, \mathbb{Q})$, since it is naturally defined algebrically.
There is an induced rational Hodge structure on the quotient

$$H^6(Y, \mathbb{Q})/\gamma^{-1}(A^{4n-2})_0$$

and we can apply once again Deligne's Lemma once again to the product map

$$H^2(Y, \mathbb{Q})^{\otimes 3} \to H^6(Y, \mathbb{Q})/\gamma^{-1}(A^{4n-2})_0 :$$

after composing with the algebraic map

$$H^2(Y, \mathbb{Q}) \to H^2(Y, \mathbb{Q})^{\otimes 3},\ \alpha \mapsto \alpha^3,$$

we deduce that $\gamma^{-1}(D) \otimes \mathbb{C}$ is an irreducible component of the set

$$Z''' := \{\delta \in H^2(Y, \mathbb{C}) \mid \delta^3 = 0 \in H^6(Y, \mathbb{C})/(\gamma^{-1}(A^{4n-2})_0 \otimes \mathbb{C})\}$$

As $\gamma^{-1}(D)$ is a rational subspace of $H^2(Y, \mathbb{Q})$, we deduce that it is a rational sub-Hodge structure of $H^2(Y, \mathbb{Q})$. □

Proof of Theorem (3.1.2). Recall that by its definition

$$\gamma : H^*(Y, \mathbb{Q}) \cong H^*(X', \mathbb{Q})$$

is compatible up to a coefficients with Poincaré duality, given by cup-products and by the isomorphisms

$$H^{4n+4}(X', \mathbb{Q}) \cong \mathbb{Q},\ H^{4n+4}(Y, \mathbb{Q}) \cong \mathbb{Q}.$$

We have already noted that $\gamma^{-1}(A^{4n-4}) \subset H^{4n-4}(Y, \mathbb{Q})$ is a sub-Hodge structure.
Let us consider the inclusion

$$A^{4n-4} \hookrightarrow H^{4n-4}(X', \mathbb{Q});$$

its Poincaré dual is the map

$$(q \circ \psi)_* : H^8(X', \mathbb{Q}) \to H^4((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q}).$$

Recall also that by the Künneth theorem, there is an isomorphism in cohomology

$$H^4((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q}) \cong A^4_{1\mathbb{Q}} \oplus A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}} \oplus A^4_{2\mathbb{Q}}.$$

Projecting onto the middle summand of the previous decomposition, we obtain a map

$$\kappa : H^4((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q}) \to A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}};$$

via γ and composition of maps, we get a morphism of Hodge structure

$$H^8(Y, \mathbb{Q}) \to \gamma^{-1}(A^2_{1\mathbb{Q}}) \otimes \gamma^{-1}(A^2_{2\mathbb{Q}}),$$

since $\gamma^{-1}(A^2_{i\mathbb{Q}}),\ i = 1, 2$ are sub-Hodge structures.
Let us restrict this morphism to the sub-Hodge structure

$$\gamma^{-1}(D)^4 = \gamma^{-1}(D^4) \subset H^8(Y, \mathbb{Q})$$

generated by $\gamma^{-1}(D)$; we will denote it π_γ. Clearly π_γ is conjugate via γ to the restriction of $\kappa \circ (q \circ \psi)_*$ to D^4.
We need now the following two lemmas:

Lemma 3.3.3. *The image of*

$$\kappa \circ (q \circ \psi)_* : D^4 \to A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$$

contains

$$Id \in Hom(A^2_{1\mathbb{Q}}, A^2_{1\mathbb{Q}}) \cong A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}$$

and

$$\phi^* = \wedge^2 \phi^t \in Hom(A^2_{1\mathbb{Q}}, A^2_{1\mathbb{Q}}) \cong A^2_{1\mathbb{Q}} \otimes A^2_{2\mathbb{Q}}.$$

We will denote

$$\Pi_\gamma = (\gamma^{-1} \otimes \gamma^{-1}) \circ (Im\, \kappa \circ (q \circ \psi)_*) \subset \gamma^{-1}(A^2_{1\mathbb{C}}) \otimes \gamma^{-1}(A^2_{2\mathbb{C}})$$

the image of π_γ.

Lemma 3.3.4. *1. Consider $u \in \gamma^{-1}(A^2_{1\mathbb{Q}}) \otimes \gamma^{-1}(A^2_{2\mathbb{Q}})$ as an element of*

$$Hom(\gamma^{-1}(A^2_{2\mathbb{Q}})^*, \gamma^{-1}(A^2_{1\mathbb{Q}})).$$

By the word non degenerate, we shall mean that u represents an isomorphism. Now, the assertion is that the generic element of Π_γ is non-degenerate.

2. The $\mathbb{Q}$-vector subspace Π'_γ of $End(\gamma^{-1}(A^{2*}_{2\mathbb{Q}}))$ generated by elements $u^{-1} \otimes v$, $u, v \in \Pi'_\gamma$, u non-degenerate, consists of those Hodge classes in $End(\gamma^{-1}(A^{2*}_{2\mathbb{Q}}))$.

We shall assume these two Lemmas, whose proof we postpone.
A consequence of the two Lemma is that the Hodge structure of $\gamma^{-1}(A^{2*}_{2\mathbb{Q}})$ admits an endomorphism conjugate to $\phi^*_T = \wedge^2\phi^t$, hence, for duality, the Hodge structure of $\gamma^{-1}(A^2_{2\mathbb{Q}})$ admits a morphism conjugate to $\wedge^2\phi$.
We conclude now as in Section 2.2.2 of Chapter 2. By the above observations, either $\gamma^{-1}(A^2_{2\mathbb{Q}})$ has a trivial Hodge structure or it does not contain any Hodge class. By a Hodge index argument we can exclude the first case.
Working symmetrically with $A^2_{1\mathbb{Q}}$, we conclude similarly that the Hodge structure on $\gamma^{-1}(A^2_{1\mathbb{Q}})$ does not contain any Hodge class. Thus it follows from Corollary (3.3.2) that the only degree 2 Hodge classes on Y are contained in $\gamma^{-1}(D)$. Looking at the intersection form

$$q_d = \int_Y d^{4n}\alpha\beta,\ \alpha, \beta \in \gamma^{-1}(A^2_{1\mathbb{Q}}), d \in \gamma^{-1}(D),$$

we conclude that it is zero on $\gamma^{-1}(A^2_{1\mathbb{Q}})$ since tha same is true for D and $A^2_{1\mathbb{Q}}$ on X'. For no degree 2 Hodge class d on Y, the sub-Hodge structure $\gamma^{-1}(A^2_{1\mathbb{Q}}) \subset H^2(Y, \mathbb{Q})$ can be polarized by q_d. Thus, Y is not projective.

∎

Proof of Lemma (3.3.3). We use the same notations as in Proposition (3.2.16).
We first reduce to the case $X' = X$.
Using Lemma (3.2.17) we have that for any Zariski open set $U \subset K_0 \times \hat{K}_0$, the restriction map

$$rest_U : H^4(K_0 \times \hat{K}_0, \mathbb{Q}) = H^4((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q}) \to H^4(U, \mathbb{Q})$$

is an isomorphism. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} rest_U : D^4 \subset H^8(X', \mathbb{Q}) & \longrightarrow & H^8(X'_U, \mathbb{Q}) \\ \downarrow{\scriptstyle (q\circ\psi)_*} & & \downarrow{\scriptstyle (q\circ\psi)_{|U*}} \\ rest_U : H^4((T/\pm Id) \times (\hat{T}/\pm Id), \mathbb{Q}) & \longrightarrow & H^4(U, \mathbb{Q}). \end{array} \tag{3.21}$$

Now, Proposition (3.2.16) says that the meromorphic map ψ is well defined on a Zariski open set X'_U as above. We thus have anoter commutative diagram

$$\begin{array}{ccc} Hdg^2(X)^4_{|X_U} & \xrightarrow{\psi_U^*} & D^4_{|X'_U} \\ \downarrow{\scriptstyle q_{|U*}} & & \downarrow{\scriptstyle (q\circ\psi)_{|U*}} \\ H^4(U,\mathbb{Q}) & \xrightarrow{\cong} & H^4(U,\mathbb{Q}), \end{array}$$

where $q_U, (q\circ\psi)_{|U}$ denote the restrictions of $q, q\circ\psi$ to X_U, X'_U respectively. We are using the fact that degree 2 Hodge classes on X, restricted to X_U, pull-back via ψ_U to degree 2 Hodge classes on X', restricted to X'_U, which follows from the fact that ψ is meromorphic and the considerations at the end of Section 3.1.
Writing for X the same diagram as (3.21), we conclude that, indeed, if we can prove it for X, the result for X' readily follows.
Let us look now at the following diagram:

$$\begin{array}{ccc} \bar{q}:\mathbb{P}(E)_0\times_{T_0\times\hat{T}_0}\mathbb{P}(E_\phi)_0 & \longrightarrow & T_0\times\hat{T}_0 \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ q: X_0 & \longrightarrow & K_0\times\hat{K}_0 \end{array}$$

where the lower indices 0 denote the restrictions of the projective bundle over $T_0\times\hat{T}_0$ and the vertical maps denoted by e are the quotient maps and X_0 is the smooth part of the quotient Q.
Arguing as above, we can replace X by X_0 and then X_0 by its étale cover $\mathbb{P}(E)_0\times_{T_0\times\hat{T}_0}\mathbb{P}(E_\phi)_0$.
Thus the result will follow from the following facts.
let Σ, Σ_ϕ be the two divisors of (3.17) and let σ, σ_ϕ be their cohomology classes. We have relations

$$\begin{aligned} \bar{q}_*(e^*(s^3 s_\phi)) &= -16 Id \in Hom(H^2(T_0,\mathbb{Q}), H^2(T_0,\mathbb{Q})) = H^2(T_0,\mathbb{Q})\otimes H^2(\hat{T}_0,\mathbb{Q}), \quad (3.22)\\ \bar{q}_*(e^*(s_\phi^3 s)) &= -16\phi^* \in Hom(H^2(T_0,\mathbb{Q}), H^2(T_0,\mathbb{Q})) = H^2(T_0,\mathbb{Q})\otimes H^2(\hat{T}_0,\mathbb{Q}). \end{aligned}$$

The formula is deduced as follows.
Let s_1, s_2 be the classes of divisors Σ_1, Σ_2 of $\mathbb{P}(E)_0\times_{T_0\times\hat{T}_0}\mathbb{P}(E_\phi)_0$ given by the decomposition $E=\mathcal{P}\oplus\mathcal{P}^{-1}$ and similarly let s_1^ϕ, s_2^ϕ be the classes of divisors $\Sigma_1^\phi, \Sigma_2^\phi$ of $\mathbb{P}(E)_0\times_{T_0\times\hat{T}_0}\mathbb{P}(E_\phi)_0$ given by the decomposition $E_\phi=\mathcal{P}_\phi\oplus\mathcal{P}_\phi^{-1}$. We have

$$e^*(s) = s_1+s_2,\ e^*(s_\phi) = s_1^\phi + s_2^\phi.$$

Let h, h_ϕ be respectively $c_1(\mathcal{O}_{\mathbb{P}(E)}(1)),\ C_1(\mathcal{O}_{\mathbb{P}(E_\phi)})$ or rather their pull-backs to $\mathbb{P}(E)_0\times_{T_0\times\hat{T}_0}\mathbb{P}(E_\phi)_0$. Let p, p_ϕ be the classes $c_1(\mathcal{P}),\ c_1(\mathcal{P}_\phi)$. We have the following relations

$$\begin{aligned} s_1 &= -\bar{q}^*(p)+h,\ s_2=\bar{q}^*(p)+h\\ s_1^\phi &= -\bar{q}^*(p_\phi)+h,\ s_2=\bar{q}^*(p_\phi)+h \end{aligned}$$

Thus
$$e^*(s) = 2h,\ e^*(s_\phi) = 2h_\phi,$$
and
$$e^*(s^3 s_\phi) = 16h^3 h_\phi,\ e^*(s_\phi^3 s) = 16h_\phi^3 h.$$
Applying $\bar{q}_*$, we conclude that
$$\bar{q}_*(e^*(s^3 s_\phi)) = -16c_2(E),\ \bar{q}_*(e^*(s_\phi^3 s)) = -16c_2(E_\phi).$$
As $E = \mathcal{P} \oplus \mathcal{P}^{-1}$ and $E_\phi = \mathcal{P}_\phi \oplus \mathcal{P}_\phi^{-1}$, it follows that
$$c_2(E) = -p^2, c_2(E_\phi) = -p_\phi^2.$$
We have that p identifies to
$$Id \in Hom(H^1(T,\mathbb{Q}), H^1(T,\mathbb{Q})) = H^1(T,\mathbb{Q}) \otimes H^1(\hat{T},\mathbb{Q}) \subset H^2(T \times \hat{T},\mathbb{Q})$$
hence p^2 identifies to
$$Id \in Hom(H^2(T,\mathbb{Q}), H^2(T,\mathbb{Q})) = H^2(T,\mathbb{Q}) \otimes H^2(\hat{T},\mathbb{Q}) \subset H^4(T \times \hat{T},\mathbb{Q}).$$
Similarly p_ϕ^2 identifies to
$$\wedge^2\phi \in Hom(H^2(T,\mathbb{Q}), H^2(T,\mathbb{Q})) = H^2(T,\mathbb{Q}) \otimes H^2(\hat{T},\mathbb{Q}) \subset H^4(T \times \hat{T},\mathbb{Q}).$$
Thus formula (3.22) is proved.

■

Proof of Lemma (3.3.4).

1. It is a consequence of Lemma (3.3.3), since Π_γ contains at least an invertible element.

2. Since Π_γ is a sub-Hodge structure of
$$\gamma^{-1}(A^2_{1\mathbb{Q}}) \otimes \gamma^{-1}(A^2_{2\mathbb{Q}}),$$
it follows that the space Π'_γ is a sub-Hodge structure of $End_{\mathbb{Q}}(\gamma^{-1}A^{2*}_{2\mathbb{Q}})$. Thus, the same is true for the subalgebra of $End_{\mathbb{Q}}(\gamma^{-1}(A^{2*}_{2\mathbb{Q}}))$ generated by Π'_γ.
On the other hand, Π'_γ is conjugate via γ^t to the corresponding subspace of $End_{\mathbb{Q}}(A^{2*}_{2\mathbb{Q}})$, defined similarly starting from $Im\ \kappa \circ (q \circ \psi)_{*|D^4}$. This last subspace is contained in the space of endomorphisms of Hodge structure of $A^{2*}_{2\mathbb{Q}}$, which has been computed to be equal to the algebra generated by $\wedge^2\phi$ (see the proof of Lemma (3.2.11)).
As $\wedge^2\phi$ is diagonalizable, this algebra tensored with $\mathbb{C}$ has no nilpotent element. It follows that Π'_γ has no nilpotent element. But as Π'_γ is a sub-Hodge structure of $End(\gamma^{-1}(A^{2*}_{2\mathbb{Q}}))$, it follows that it is pure of type $(0,0)$, i.e. is made only of Hodge classes, since elements of type $(-k,k),\ k>0$ are nilpotent.

■

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