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Generalizations of Aldous' Spectral gap Conjecture.

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Contents

1 Introduction

In 1981 Diaconis and Shahshahani proved that on complete unweighted graph the interchange process and the random walk have the same spectral gap [5]. This was the first example of what turned out to be a much more general fact. Further evidences arrived with the work of Flatto and Odlyzko [6] who proved the equality for all star graphs. In 1992 the problem arose in a conversation between David Aldous and Persi Diaconis: could this equality hold for all unweighted graphs? It was stated for the first time as an open problem in Alouds personal web page [1] and in the 1994 version of [2], since then it has been called Aldous' Spectral gap conjecture or Diaconis-Aldous conjecture. From that moment the problem received more ad more attention. In 1996 Handjani and Jungreis proved the equality for all weighted trees [9] using a recursive approach. The 2008 was the year of asymptotic version for boxes in $\mathbb{Z}^d$, with unweighted edges with Conomos and Starr [13], and Morris [12]. Recently Cesi push the algebraic approach of [5] to obtain the conjecture for all unweighted complete multipartite graphs [4]. The conjecture was finally proved in 2009 in a joint work of Caputo, Liggett and Richthammer [3] who presented a proof for all weighted graphs. This article is surveyed in section 2.

In section 3 we explore the frontiers of Aldous' conjecture by exploring possible generalization. First we show, by means of numeric counterexamples, that the Spectral gap scenario fails for other quantities such as Log-Sobolev and modified Log-Sobolev. So that we cannot extend the "reductive" approach of the conjecture. Another natural way of generalizing the conjecture is by considering more "complicated" models. We introduce two models, $k-$Deck and Block Shuffle. In $k-$Deck we increase the number of particles on every node of the graph from 1 to an arbitrary integer k. Heuristically speaking, we consider every node as a deck of k cards. This process generalizes the interchange process, that is the particular case $k = 1$. For this process we state an analogue of the Aldous' conjecture and we give a proof in a very special case. An higher degree of generalization is obtained through the model we call Block Shuffle. We lose the structure of the underlying graph. The dynamics consists of shuffles of entires blocks, identified as subsets of a given set V, which are updated each with its own rate. Namely we have an application

$$\alpha : \begin{array}{ccc} \mathcal{P}(V) \setminus \{\emptyset\} & \longrightarrow & [0, +\infty) \\ A & \longmapsto & \alpha_A \end{array}$$

where $\mathcal{P}(V)$ is the power set of V. We consider subsets A such that $\alpha_A > 0$. This defines the rates of Poisson' clocks. When the clock with rate α_A "rings", the corresponding subset equilibrates (i.e. the cards sitting at the vertices of A are fully shuffled). We observe that this model generalizes both interchange process and $k-$Deck. We state an analogue of the Aldous' conjecture for Block Shuffle model. Finally in section 4 we prove the conjecture for a special class of Block Shuffle, that we call *simple*. The particularity of this class is the possibility of reducing the Block structure to a binary structure by means of suitably simple reductions. This allows us, once we remove all "non necessary" points, to obtain a binary block process, that we are able to deal with, using the techniques of [3].

2 Aldous' Spectral gap conjecture

The conjecture is about the random walk process and the interchange process on a weighted graph. From now on our graphs will be connected, because we want the Markov chains to be irreducible. Moreover every graph will be undirected, that is

$$c_{xy} = c_{yx} \tag{1}$$

for every $xy \in E$, where c_{xy} is the weight of edge xy. Such Markov chains are reversible with respect to the uniform distribution on Ω. Then if we compute the quadratic form of the infinitesimal generator we find out that

$$-\mathcal{L} \geq 0. \tag{2}$$

The spectrum of $-\mathcal{L}$ is of the form

$$\mathrm{Spec}\,(-\mathcal{L}) = \{\lambda_i : i = 0, ..., |\Omega| - 1\}$$

with

$$0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{|\Omega|-1}.$$

2.1 The processes

Random Walk The random walk on a graph $G = (V, E)$ is the Markov chain in which a particle in vertex $x \in V$ jump to $y \neq x$ with rate c_{xy}. Its space state is $\Omega^{RW} = V = \{1, 2, ..., n\}$. The generator is defined by

$$\mathcal{L}^{RW} f(x) = \sum_{x \neq y} c_{xy}(f(y) - f(x)) \tag{3}$$

for $f : V = \Omega^{RW} \to \mathbb{R}$ and $x \in V$.
Moreover $-\mathcal{L}^{RW}$ is nonnegative semi definite and symmetric then it has $|\Omega^{RW}| = n$ nonnegative eigenvalues and positive spectral gap $\lambda_1^{RW} > 0$.

Interchange A state in interchange process is an assignment of $|V| = n$ labeled particles to the n vertices of the graph G, so that any vertices is occupied by exactly one labeled particle. We identify a state with an element $\eta \in S_n$ that is the symmetric group of the n elements permutations. Formally the associated Markov chain is the one in which with rate c_{xy} occurs a transition from state η to stateη^{xy} that is an "interchange" of the particles at vertices x and y, to be more precise $\eta^{xy} = \eta \circ (xy)$, so that the state space of the associated Markov chain is $\Omega^{IP} = S_n$. The generator is defined by

$$\mathcal{L}^{IP} f(\eta) = \sum_{xy \in E} c_{xy}(f(\eta^{xy}) - f(\eta)) \tag{4}$$

with $f : \Omega^{IP} \to \mathbb{R}$ and $\eta \in \Omega^{IP}$. The operator $-\mathcal{L}^{IP}$ is symmetric and positive semidefinite, then it has $|\Omega^{IP}| = n!$ nonnegative eigenvalues and positive spectral gap $\lambda_1^{IP} > 0$.

Conjecture 1. *For all weighted graph G, the interchange process and the random walk process have the same spectral gap,*

$$\lambda_1^{RW}(G) = \lambda_1^{IP}(G). \tag{5}$$

We will refer to the label of the particle at x as η_x, and to the position the particle labeled i as $\xi_i = \xi_i(\eta)$.
One can obtain the random walk as a subprocess of the interchange process following just the particle labeled 1. Thus the inequality

$$\lambda_1^{RW} \geq \lambda_1^{IP}. \tag{6}$$

is easy to prove.

2.2 Novelties

There are two main novelties introduced in [3], that made possible the proof of Aldous' conjecture. The first is a generalized electric network reduction and the second is a tricky inequality involving Dirichlet forms, called Octopus Inequality.

Network reduction Given a weighted graph $G = (V, E)$ we can think about it as a network and the weights on the edges as conductances. Consider now a vertex $x \in V$, the reduced network obtained by removing x gives a new graph G_x with vertices set $V_x := V \setminus \{x\}$ and edges set $E_x := \{yz \in E : y, z \neq x\}$. The conductance in the new graph are such that $\widetilde{c}_{xy} \geq c_{xy}$ defined by

$$\widetilde{c}_{yz} = c_{yz} + c_{yz}^{*,x} \tag{7}$$

where

$$c_{yz}^{*,x} := \frac{c_{xy}c_{xz}}{\sum_{w \in V_x} c_{xw}}. \tag{8}$$

The next result is about the behavior of the Spectral gap of the Random walk with respect to the reduction of the undelying graph.

Proposition 2.1. *The spectral gap of the random walk do not decrease when the underlying graph is reduced,*

$$\lambda_1^{RW}(G_x) \geq \lambda_1^{RW}(G).$$

Octopus Inequality In this subsection we introduce the Octopus inequality. From now on let $\mathbb{E}\left[\,\cdot\,\right]$ denote the expectation with respect to the uniform probability measure on S_n, and the gradient defined by

$$\nabla_{xy} f = f(\eta^{xy}) - f(\eta) \text{ equivalentely } \nabla_b f = f(\eta^b) - f(\eta).$$

Using this notation we have the following theorem.

Theorem 2.2 (Octopus Inequality)**.** *For any weighted graph Gon $|V| = n$ vertices, for every $x \in V$ and $f : S_n \to \mathbb{R}$:*

$$\sum_{y \in V_x} c_{xy} \mathbb{E}\left[\,(\nabla_{xy} f)^2\,\right] \geq \sum_{yz \in E_x} c^{*;x}_{yz} \mathbb{E}\left[\,(\nabla_{yz} f)^2\,\right] \tag{9}$$

It is important to note that inequality (9) holds for every choice of non-negative weights $\{c_b\}_{b \in E}$.

2.3 Proof of the conjecture

If with set $\mathcal{H}_{\Omega^{IP}} := \{f \text{ such that } f : \Omega^{IP} \to \mathbb{R}\}$, we have

$$\lambda_1^{IP}(G) = \inf_{\mathcal{H}_{\Omega^{IP}}} \frac{-\mathbb{E}\left[\,f \mathcal{L}^{IP}(f)\,\right]}{\mathrm{Var}_{IP}(f)} = \inf_{\mathcal{H}_{\Omega^{IP}}} \frac{\mathcal{E}^{IP}(f)}{\mathrm{Var}_{IP}(f)}.$$

where $\mathrm{Var}_{IP}(f) = \mathbb{E}\left[\,f^2\,\right] - (\mathbb{E}\left[\,f\,\right])^2$. We already showed that

$$\mathrm{Spec}\left(-\mathcal{L}^{RW}\right) \subset \mathrm{Spec}\left(-\mathcal{L}^{IP}\right),$$

in order to understand better the relation between these two spectra we define

$$\begin{aligned}\mathcal{H} &= \{f \in \mathcal{H}_{\Omega^{IP}} : \mathbb{E}\left[\,f|\xi_i\,\right] = 0 \text{ for all } i \in V\} = \\ &= \{f \in \mathcal{H}_{\Omega^{IP}} : \mathbb{E}\left[\,f|\eta_x\,\right] = 0 \text{ for all } x \in V\}\end{aligned}$$

the identity holds because, for $\eta \in S_n$ such that $\xi_i(\eta) = x$,

$$\mathbb{E}\left[\,\cdot|\xi_i\,\right](\eta) = \mathbb{E}\left[\,\cdot|\xi_i = x\,\right] = \mathbb{E}\left[\,\cdot|\eta_x = i\,\right] = \mathbb{E}\left[\,\cdot|\eta_x\,\right](\eta).$$

Lemma 2.3. *If $f \in \mathcal{H}_{\Omega^{IP}}$ is an eigenfunction with eigenvalue λ, that is*

$$\mathcal{L}^{IP} f = -\lambda f\,,$$

if we define $g(x) = \mathbb{E}\left[\,f\,|\,\xi_1\,\right](x)$ and $g : V \to \mathbb{R}$, we have

$$\mathcal{L}^{RW} g = -\lambda g\,.$$

We are able to split the problem, splitting the spectrum. From the variational principle defining the Spectral gap it follows

$$\lambda_1^{IP}(G) = \min\{\mu_1^{IP}(G),\, \lambda_1^{RW}(G)\}. \tag{10}$$

if we define

$$\mu_1^{IP}(G) = \inf_{\mathcal{H}} \left\{ \frac{\mathcal{E}^{IP}(f)}{\mathrm{Var}_{IP}(f)},\ \mathrm{Var}_{IP}(f) \neq 0 \right\}.$$

The conjecture can now be reformulated as

$$\mu_1^{IP}(G) \geq \lambda_1^{RW}(G).$$

Let us show a property of μ_1^{IP}, this is an halfway step to the Conjecture proof.

Proposition 2.4. *For an arbitrary weighted graph G*

$$\mu_1^{IP}(G) \geq \max_{x \in V} \lambda_1^{IP}(G_x) \tag{11}$$

We can now go through the proof.

Theorem 2.5. *For all weighted graph G, the interchange process and the random walk process have the same spectral gap,*

$$\lambda_1^{RW}(G) = \lambda_1^{IP}(G) \tag{12}$$

Proof. We go through the proof by induction on the number of elements of the vertices set V.

$(n = 2)$ For $n = 2$ we have a trivial graph G, made of just one weighted edge (xy). In this case (12) holds because the random walk process coincides with the interchange process as $2-$state Markov chain, when $n = 2$.

$(n - 1 \Rightarrow n)$ Suppose now that (12) holds for all $n - 1$ vertices weighted graph G'. This means that in particular holds for G_x. Now

- From Proposition 2.4
$$\mu_1^{IP}(G) \geq \max_{x \in V} \lambda_1^{IP}(G_x)$$
- From inductive hypotesis
$$\max_{x \in V} \lambda_1^{IP}(G_x) = \max_{x \in V} \lambda_1^{RW}(G_x)$$
- From Proposition 2.1
$$\max_{x \in V} \lambda_1^{RW}(G_x) \geq \lambda_1^{RW}(G)$$

Thus
$$\mu_1^{IP}(G) \geq \lambda_1^{RW}(G)$$

and the conjecture holds for any weighted graph. ∎

3 Generalizations

3.1 Spectral Gap scenario fails for other quantities

We present two numerical counterexample in order to show that, in the case of a complete graph with 3 nodes, the Log-Sobolev constant and the modified Log-Sobolev constant of interchange process and random walk process are different.

3.2 $k-$Deck Generalization

The first generalization we want to suggest is through the process we called $k-$Deck. Heuristically imagine a graph with a deck of k cards on every vertex. When an edge "rings" we put together the $2k$ cards of the two decks at the ends of the edge, then we shuffle them and once they are randomly rearranged, we put back the first k on one vertex an the other k on the other. Let us introduce better this model.

The model We work with an undirected graph $G = (V, E)$. We refer to the elements of V, as *nodes* or *decks* labeled from 1 to n. Now, on every node, there is a set of k elements, $k \in \mathbb{N}$ fixed, that we call *vertices* or *cards*. We have to think about this system as a graph with a deck of k cards on every node. We associate to the elements $b \in E$, a collection of weights $\{c_b\}_{b\in E}$, with $c_b \geq 0$ for all $b \in E$, such that the skeleton graph is connected. Formally the $k-$Deck process is the Markov chain in which, if edge b connects nodes i and j, with rate c_b the $2k$ cards of decks i and j are equilibrated, then we put the first k on vertex i and the other k on vertex j.

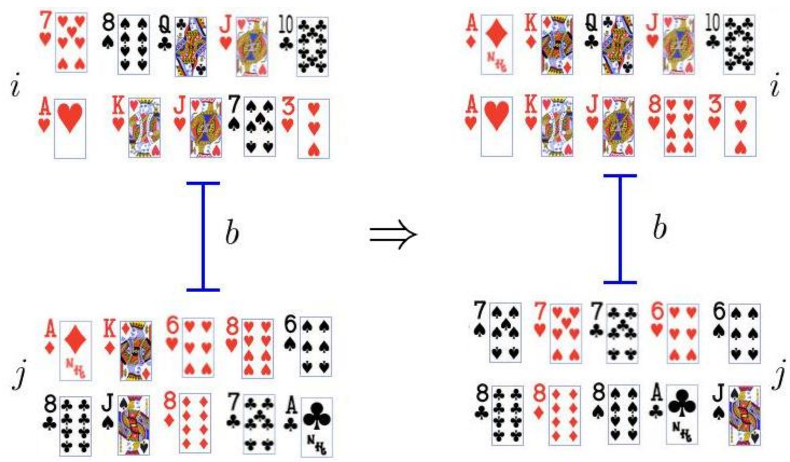

Figure 1: *Heuristic representation of $k-$Deck, here the clock on edge b has rung. In this example $k = 10$.*

The state space is $\Omega^{kD} = S_{kn}$, the group of permutations of kn elements, and its generator is

$$\mathcal{L}^{kD} = \sum_{b \in E} 2c_b(\mu_b - \mathbb{1}) \tag{13}$$

where

$$\mu_b f(\eta) = \mathbb{E}\,[\, f \,|\, \eta_m \;\; m \neq i,j \;\; \text{if } b = (i,j)\,] = \frac{1}{(2k)!} \sum_p f(\eta_{ij}^p) \tag{14}$$

where η_{ij}^p is one of the $(2k)!$ possible outcomes of the shuffling of deck on vertices i and j.
Note that the 2 multiplying the weight c_b in (13) is due in order to have an homogeneity with the case $k = 1$, corresponding to the interchange process.
The Dirichlet form is

$$\mathcal{E}^{kD}(f) = \sum_b 2c_b \mathbb{E}\left[\, (\mu_b(f))^2 - f^2 \,\right] = \sum_b 2c_b \mathrm{Var}_b\,(f)\,, \tag{15}$$

where $\mathrm{Var}_b\,(\cdot)$ is the conditional variance with respect to $\mu_b(\cdot)$. Following, for example, the card with the label 1 we find again

$$\lambda_1^{RW} \geq \lambda_1^{kD}.$$

The conjecture should be as follows.

Conjecture 2. *The $k-$Deck process and the random walk process, on all weighted graph G, have the same Spectral Gap,*

$$\lambda_1^{kD}(G) = \lambda_1^{RW}(G) \tag{16}$$

We end this section with a really special case, where we are able to prove the conjecture. Let the number of nodes be $n = 2$ and arbitrary k. For $k-$Deck we have $\mathrm{gap}^{kD} = 2c_b$, because

$$\mathcal{E}^{kD}(f) = 2c_b \mathrm{Var}_b\,(f) = 2c_b \mathrm{Var}\,(f) \tag{17}$$

Even for the random walk $\mathrm{gap}^{RW} = 2c_b$. Let start with the generator. In this special case, for $g \in \mathcal{H}_{\Omega^{RW}}$, we have

$$\begin{cases} (\mathcal{L}^{RW} g)(x) = c_b(g(y) - g(x)) \\ \\ (\mathcal{L}^{RW} g)(y) = c_b(g(x) - g(y)) \end{cases} \tag{18}$$

is now simple to compute the Dirichlet form

$$\mathcal{E}^{RW}\,(g) = -\mathbb{E}\left[\, g(\mathcal{L}^{RW} g) \,\right] = \frac{1}{2} c_b \left(g(x) - g(y)\right)^2. \tag{19}$$

On the other and for the variance of the function g we have

$$\mathrm{Var}\,(g) = \frac{1}{4}(g(x) - g(y))^2. \tag{20}$$

So $\mathrm{gap}^{RW} = 2c_b$, and the conjecture holds.

3.3 Block Shuffle Generalization

A further generalization is represented by the process we call Block Shuffle.

The model We have a set V of n elements and we consider all the subsets $\emptyset \neq A \subseteq V^{BS}$, to which we assign a weight $\alpha_A \geq 0$. We just consider the subsets A such that $\alpha_A > 0$. The Block Shuffle model is the Markov chain that with rate α_A equilibrates the particle of subset A. Therefore the generator of this process is

$$\mathcal{L}(\alpha) = \sum_{A \subseteq V^{BS}} \alpha_A(\mu_A - \mathbb{1}) \tag{21}$$

where

$$\mu_A(f) = \mathbb{E}\left[f \,\middle|\, \eta^{A^C} \right].$$

The state space is $\Omega^{BS} = S_n$ and $|\Omega^{BS}| = n!$.
If we follow only one particle in the Block Shuffle it does a random walk with weights

$$c_{yz}(\alpha) = \sum_{A:\{y,z\}\subset A} \frac{\alpha_A}{|A|}. \tag{22}$$

In fact, let be $f = g(\xi_1)$, then

$$\mathcal{L}(\alpha)g(z) = \sum_y \left(\sum_{\{y,z\}\subset A} \frac{\alpha_A}{|A|} \right) [g(y) - g(z)]. \tag{23}$$

Thus the particle jump randomly on a graph that we call $G(\alpha)$. This graph has vertices set V and edges set

$$E(\alpha) = \{yz \text{ such that } c_{yz}(\alpha) > 0\}.$$

Let $\lambda_1(\alpha)$ be the smallest non zero eigenvalue of $-\mathcal{L}(\alpha)$, the reformulation of the conjecture for Block Shuffle should be as follows.

Conjecture 3. *The Block Shuffle process and the random walk process induced by the Block Shuffle process on the graph $G(\alpha)$, have the same Spectral Gap,*

$$\lambda_1(\alpha) = \lambda_1^{RW}(G(\alpha)) \tag{24}$$

A generalization We now show that this process generalize the interchange process. Let us consider the special case of binary blocks

$$\alpha_A > 0 \Longleftrightarrow |A| = 2 \tag{25}$$

In this case we can introduce a graph structure, because we are considering just couple of linked points. Then if we think at subsets A as edges b, the generator of the process is

$$\widetilde{\mathcal{L}} = \sum_b \alpha_b(\mu_b - \mathbb{1}) \tag{26}$$

Where $\mu_b(\cdot)$ has the particular structure,

$$\mu_b(f) = \frac{1}{2}f(\eta^b) + \frac{1}{2}f(\eta). \tag{27}$$

Thus we have

$$\widetilde{\mathcal{L}}f(\eta) = \sum_b \alpha_b \left(\frac{f(\eta^b) + f(\eta)}{2} - f(\eta) \right) = \mathcal{L}^{IP} f(\eta) \tag{28}$$

that correspond to the generator of the Interchange process with $c_b = \frac{1}{2}\alpha_b$. The Block Shuffle process is also a generalization of $k-$Deck process. We introduce the notation $k_i^{(x)}$ for the $i-$th card of the deck on node x and $k^{(x)}$ for the entire deck. Let us consider a $k-$Deck process on a graph $G = (V, E)$ and weights set $\{c_b\}_{b\in E}$. We can see this process as an Block Shuffle if we take

$$V = \{k_i^{(x)} \ : \ x \in V \ , \ i = 1, ..., k\},$$

as the set of all the kN cards, if $|V| = N$ is the number of decks. As subsets $A \subset V$ we take, for every $xy \in E$,

$$A = A(x, y) = \{k_i^{(z)} \ : \ i = 1, ..., k \ \text{ and } z = x, y\}$$

with weights $\alpha_A = \alpha_{A(x,y)} = 2c_{xy}$. Let us focus on the random walk of one card. In the particular case of $k-$Deck the graph $G(\alpha)$, on which the card "walk randomly", has the particular structure of a product graph. On every node of the graph G there is a complete graph with k elements. We call this new graph $\mathcal{G}$.

We denote its vertices as pairs (x, i), where x is a node of graph G and i is a node of the complete graph. Thus we have

$$\mathcal{V} = \{(x, i) : x \in V \text{ and } i = 1, ..., k\}.$$

The graph $\mathcal{G}$ has many edges. Fix the position of the card in (x, i). Now it can jump in position (y, j) for every y neighboring with x in graph G, for every $j = 1, ..., k$, thus

$$\mathcal{E} = \{(x, i)(y, j) : i, j = 1, ..., k \text{ and } xy \in E\}.$$

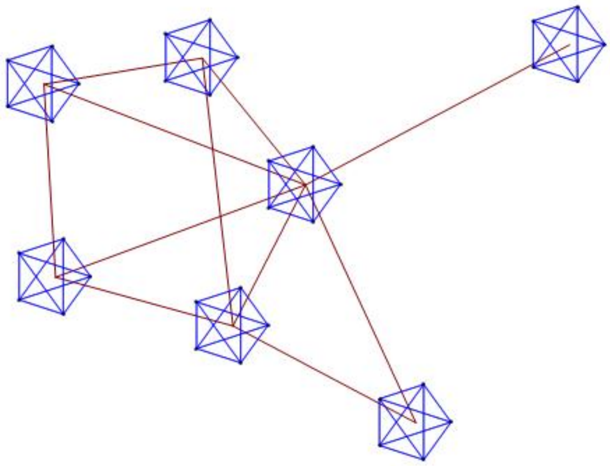

Figure 2: *The graph $\mathcal{G}$. In this example $k = 5$.*

Following (22) the weights are

$$c_{(x,i)(y,j)} = \begin{cases} \dfrac{1}{k}\displaystyle\sum_z c_{xz} = \dfrac{c(x)}{k} & \text{if } x = y \\[2ex] \dfrac{c_{xy}}{k} & \text{if } x \neq y \end{cases}$$

Now a contradiction seems to arise. If we consider the process as a Block Shuffle, conjecture 3 tell us that the gap of this process should be equal to the gap $\lambda_1^{RW}(\mathcal{G})$ of the random walk on graph $\mathcal{G}$. Nevertheless if we take the process as $k-$Deck, following conjecture 2, the gap should be equal to $\lambda_1^{RW}(G)$, the gap of the random walk on graph G. Next proposition make clear that there is no ambiguity.

Proposition 3.1.

$$\lambda_1^{RW}(\mathcal{G}) = \lambda_1^{RW}(G)$$

Proof. We define the set $\mathcal{S}$ of the functions defined on $\mathcal{V}$ and symmetric on the complete graphs, as

$$\mathcal{S} = \{f \ : \ f(x,i) = f(x,j) \text{ for every } x \in G \text{ and } i = 1, ..., k\},$$

then the orthogonal set $\mathcal{S}^\perp$ is

$$\mathcal{S}^\perp = \{f \ : \ \sum_i f(x,i) = 0 \text{ for every } x\}.$$

In fact take $g \in \mathcal{S}$ and $f : \mathcal{V} \to \mathbb{R}$, if we denote $\mathbb{E}\left[\,\cdot\,|\,k^{(x)}\,\right]$ the conditional expectation of being on node x, then

$$\mathbb{E}\left[\,fg\,\right] = \mathbb{E}\left[\mathbb{E}\left[\,fg\,|\,k^{(x)}\,\right]\right] = \mathbb{E}\left[g\mathbb{E}\left[\,f\,|\,k^{(x)}\,\right]\right] = 0$$

if and only if

$$\frac{1}{k}\sum_i f(x,i) = 0.$$

Note that if $f \in \mathcal{S}$, then

$$\mathcal{L}f(x,i) = \sum_{(y,j)} c_{(x,i)(y,j)}[f(y,j) - f(x,i)] = \sum_{(y,j)} c_{(x,p)(y,j)}[f(y,j) - f(x,p)] = \mathcal{L}f(x,p)$$

for every $p = 1, ..., k$, this follow by how we defined the weights on the graph, independently from the position on the complete graph. So $\mathcal{LS} \subset \mathcal{L}$, thus we can find the gap of the random walk on $\mathcal{G}$ as the minimum of two quantities,

$$\lambda_1^{RW}(\mathcal{G}) = \min\{\lambda_1^{\mathcal{S}}, \lambda_1^{\mathcal{S}^\perp}\}.$$

Since we defined $\mathcal{S}$ to be the set of symmetric function on the complete graphs, then $\lambda_1^{\mathcal{S}} = \lambda_1^{RW}(G)$. Now take $f \in S^\perp$, we have

$$\begin{aligned}
\mathcal{L}f(x,i) &= \sum_{(y,j)} c_{(x,i)(y,j)}[f(y,j) - f(x,i)] = \\
&= \frac{1}{k}\sum_j\sum_z c_{xz}[f(y,j) - f(x,i)] + \frac{1}{k}\sum_y\sum_j c_{xy}[f(y,j) - f(x,i)] = \\
&= -c(x)f(x,i) - c(x)f(x,i) = -2c(x)f(x,i)
\end{aligned}$$

then the eigenvalues of $-\mathcal{L}$ in $\mathcal{S}^\perp$ are

$$\{2c(x)\ :\ x \in G\}.$$

In particular,

$$\lambda_1^{\mathcal{S}^\perp} = \min_x 2c(x) = 2c^*.$$

The proof is complete if we prove $\lambda_1^{RW}(G) \leq 2c^*$. To this end let $\mathbb{1}_x$ be the indicator function of node x, then

$$\lambda_1^{RW}(G) = \inf\left\{\frac{\mathcal{E}(f)}{\mathrm{Var}\,(f)}\right\} \leq \frac{\mathcal{E}(\mathbb{1}_x)}{\mathrm{Var}\,(\mathbb{1}_x)}$$

where

$$\mathrm{Var}\,(\mathbb{1}_x) = \frac{1}{N}\left(1 - \frac{1}{N}\right)$$

and

$$\begin{aligned}\mathcal{E}(\mathbb{1}_x) &= -\mathbb{E}\left[\,\mathbb{1}_x\mathcal{L}\mathbb{1}_x\,\right] = -\frac{1}{N}\sum_{y,z}\mathbb{1}_x(y)c_{yz}(\mathbb{1}_x(z)-\mathbb{1}_x(y)) \\ &= -\frac{1}{N}\sum_{z}c_{yz}(\mathbb{1}_x(z)-1) = \frac{1}{n}c(x)\end{aligned}$$

then

$$\lambda_1^{RW}(G) \leq \frac{\frac{1}{N}c(x)}{\frac{1}{N}\left(1-\frac{1}{N}\right)}c(x) = \frac{N}{(N-1)}c(x) \leq 2c(x)$$

taking the minimum in x we have

$$\lambda_1^{RW}(G) \leq 2c^*.$$

■

3.4 Please not too general

The most general model we can think on S_n is what we call random permutations. We put weights on every permutation. The only restriction is

$$c_\pi = c_{\pi^{-1}}$$

for reversibility. The generator is, for $f : S_n \to \mathbb{R}$

$$\mathcal{L}^{RP}f(\zeta) = \sum_{\pi\in S_n} c_\pi(f(\zeta^\pi)-f(\zeta)) \tag{29}$$

where $\zeta^\pi = \pi\circ\zeta$. We can think to exend the conjecture to this model, but now consider the following counterexample.

Counterexample. Consider an undirected graph $G=(V,E)$ that is $n-$cycle, i.e. $b\in E \iff b=(i,i+1)$, where this sum, and all the following others, are taken modulo n. Let us denote $\gamma_i : S_n \to S_n$ as the rotation of i elements of this system, i.e. for $\zeta\in S_n$ we define

$$(\gamma_i\circ\zeta)_k = \zeta_{i+k}$$

note that, by this definition, γ_0 =id.
We now put weights on the permutations in this way

$$c_\pi = \begin{cases} \dfrac{1}{n} & \text{if } \pi=\gamma_i \ \text{ for } i=0,...,n-1 \\ \\ 0 & \text{otherwise}\end{cases} \tag{30}$$

Note that these assumptions follow the constraint, in fact for every i we have $c_{\gamma_i} = n^{-1} = c_{\gamma_{n-i}} = c_{(\gamma_i)^{-1}}$. Let us now compute the gap of the random walk process and the random permutations process on this graph.

(RW) The gap for the random walk process is $\lambda_1^{RW}(G) = 1$. Because in one step the system equilibrates.

(RP) For random permutations model, let consider the function $\tilde{f}(\zeta) = |\xi_1 - \xi_2|$, where ξ_1 and ξ_2 are the positions of particle 1 and 2. The function $\tilde{f}$ has, in general, variance non zero but in this special case $\tilde{f}(\zeta \circ \gamma_i) = |\xi_1 + i - (\xi_2 + i)| = |\xi_1 - \xi_2|$ for all $i = 0, ..., n-1$. So that

$$\lambda_1^{RP}(G) = 0.$$

Thus an analogous of Aldous' conjecture cannot hold for general random permutations process. And Block Shuffle seems to be the furthermost we can go.

4 Simple Block Shuffle

In this section we present a particular class of Block Shuffle for which we can prove the conjecture. In this class there are special vertices.

Definition 4.1 (Simple vertex)**.** *We define a vertex as simple if there exist an unique set* $A^* \subset V$ *such that*

(i) $\alpha_{A^*} > 0$

(ii) $x \in A^*$.

If we have to reduce the system, the correct redefinition of weights is

$$\alpha_B^{(x)} = \begin{cases} \alpha_B & B \neq A^*, A^* \setminus \{x\} \\ \alpha_{A^*} + \alpha_B & B = A^* \setminus \{x\} \\ 0 & B = A^* \end{cases} \tag{31}$$

if A^* is the unique set containing the simple vertex x.
When we reduce the system in a simple vertex x it holds an Octopus like inequality.

Theorem 4.2. *If x is a simple vertex for a Block Shuffle process determined by weights $\alpha = \{\alpha_A\}_{A \subset V}$, then*

$$\mathbb{E}\left[\mathcal{E}(f, \alpha^{(x)})\right] \leq \mathcal{E}(f, \alpha).$$

Proof. The Dirichlet form of the Block Shuffle process, is

$$\mathcal{E}\,(f, \alpha) = -\mathbb{E}\,[\,f(\mathcal{L}(\alpha)f)\,] = -\sum_{A \subset V} \alpha_A \mathbb{E}\left[\,\mu_A(f)^2 - f^2\,\right] = \sum_{A \subset V} \alpha_A \mathrm{Var}_A\,(f)\,.$$

For the reduced system we have

$$\mathcal{E}(f, \alpha^{(x)}) = \mathcal{E}(f, \alpha) - \alpha_{A^*}\mathbb{E}\,[\,\mathrm{Var}_{A^*}\,(f)\,] + \alpha_{A^*}\mathbb{E}\left[\,\mathrm{Var}_{A^* \setminus \{x\}}\,(f)\,\right] \tag{32}$$

We have to proof

$$\mathbb{E}\left[\operatorname{Var}_{A^*\setminus\{x\}}(f)\right] \leq \mathbb{E}\left[\operatorname{Var}_{A^*}(f)\right]. \tag{33}$$

This inequality follows immediately from the definitions of conditional variance

$$\operatorname{Var}_{A^*}(f) = \mathbb{E}_{A^*}[\operatorname{Var}_{A^*}(f \mid x)] + \operatorname{Var}_{A^*}(\mathbb{E}_{A^*}[f \mid x])$$

taking the expectations, and from the nonnegativity of variance, $\operatorname{Var}(\cdot) \geq 0$,

$$\mathbb{E}\left[\operatorname{Var}_{A^*}(f)\right] \geq \mathbb{E}\left[\mathbb{E}_{A^*}[\operatorname{Var}_{A^*}(f \mid x)]\right] = \mathbb{E}\left[\operatorname{Var}_{A^*\setminus\{x\}}(f)\right].$$

And the inequality is proved. ∎

4.1 The theorem

Let us define the class of simple Block Shuffle.

Definition 4.3 (Simple Block Shuffle)**.** *We define a Block Shuffle process to be simple if it can be reduced to a binary Block shuffle by means of successive reductions of simple vertices.*

For this class of Block Shuffle we can prove conjecture 3.

Theorem 4.4. *If the Block Shuffle defined by the weights $\{\alpha_A\}_{A\subset V}$ is simple, then*

$$\lambda_1(\alpha) = \lambda_1^{RW}(\alpha)$$

Proof. We define

$$\mathcal{H} = \{f \;:\; \mathbb{E}[f \mid \eta_x] = 0 \;\forall\, x\}$$

where $\mathbb{E}[\cdot \mid \eta_x]$ is the conditional expectation with respect to the reduced system in x. If we define

$$\mu_1(\alpha) = \inf_{\mathcal{H}} \left\{ \frac{\mathcal{E}(f,\alpha)}{\operatorname{Var}(f)} \right\}$$

then the Spectral gap of the Block Shuffle process can be found as the minimum of two quantities

$$\lambda_1(\alpha) = \min\{\mu_1(\alpha), \lambda_1^{RW}(\alpha)\}.$$

The Block Shuffle is simple, let us define $\{x_1, x_2, ..., x_N\}$ as the set of all the simple vertices. By Theorem 4.2 if we take $f \in \mathcal{H}$ and we fix (x_1),then

$$\begin{aligned}
\mathcal{E}(f,\alpha) &\geq \mathbb{E}\left[\mathcal{E}(f,\alpha^{(x_1)})\right] \geq \\
&\geq \lambda_1(\alpha^{(x_1)})\mathbb{E}\left[\operatorname{Var}(f \mid \eta_x)\right] = \\
&= \lambda_1(\alpha^{(x_1)})\operatorname{Var}(f)
\end{aligned}$$

thus $\mu_1(\alpha) \geq \lambda_1(\alpha^{(x_1)})$, furthermore

$$\lambda_1(\alpha) \geq \min\{\lambda_1(\alpha^{(x_1)}), \lambda_1^{RW}(G(\alpha))\}.$$

Let us now consider $\lambda_1(\alpha^{(x_1)})$, we have that

$$\lambda_1(\alpha^{(x_1)}) = \min\{\mu^1(\alpha^{(x_1)}), \lambda_1^{RW}(G(\alpha^{(x_1)}))\}$$

with analogous arguments, if we reduce the system in (x_2), we find the bound

$$\mu^1(\alpha^{(x_1)}) \geq \lambda_1(\alpha^{(x_1)(x_2)}).$$

Iterating we obtain

$$\lambda_1(\alpha) \geq \min\left\{\lambda_1^{RW}(\alpha), \lambda_1^{RW}(\alpha^{(x_1)}), \lambda_1^{RW}(\alpha^{(x_1)(x_2)}), ..., \lambda_1^{RW}(\alpha^{(x_1)(x_2)\cdots(x_N)})\right\}.$$

Now, consider the jump rate of one particle in the reduced system, for $x \in \{x_1, ..., x_N\}$ we define

$$\widetilde{c}_{yz} = c_{yz}(\alpha^{(x)}) = \sum_{\substack{\{y,z\}\in B \\ x\notin B}} \frac{\alpha^{(x)}{}_B}{|B|}. \tag{34}$$

where $\alpha_B^{(x)}$ is defined by (31). Let us define A^* as the unique set containing x. We assume $\{y, z\} \in A^*$, otherwise the rates are unchanged, and $|A^*| = k$. Following these assumptions we prove that

$$c_{yz}(\alpha^{(x)}) = c_{yz} + \frac{c_{xz}c_{yx}}{\sum_{w\in A} c_{wx}}.$$

For the following sums we introduce the notation

$$\sum\{\cdots\} \quad \text{for} \quad \sum_{\cdots}$$

We have

$$\begin{aligned}
c_{yz}(\alpha^{(x)}) &= \sum\{x \notin B,\, \{y,z\} \in B\}\, \frac{\alpha_B^{(x)}}{|B|} = \sum\{x \notin B,\, \{y,z\} \in B\}\, \frac{\alpha_B}{|B|} = \\
&= \frac{\alpha_{A^*}}{k-1} + \sum\{x \notin B,\, \{y,z\} \in B,\, B \neq A^* \setminus \{x\}\}\, \frac{\alpha_B}{|B|} = \\
&= \frac{\alpha_{A^*}}{k-1} - \frac{\alpha_{A^*}}{k} + \sum_{\{y,z\}\in B} \frac{\alpha_B}{|B|} = \\
&= c_{yz} + \frac{\alpha_{A^*}}{k-1} - \frac{\alpha_{A^*}}{k}.
\end{aligned} \tag{35}$$

Remains to prove

$$\frac{\alpha_{A^*}}{k-1} - \frac{\alpha_{A^*}}{k} = \frac{c_{xz}c_{yx}}{\sum_{w\in A} c_{wx}}.$$

Notice that

- $c_{xy} = c_{xz} = \frac{\alpha_{A^*}}{k}$
- $\sum_{w \in A} c_{wx} = \frac{\alpha_{A^*}}{k}(k-1)$

then
$$\frac{\alpha_{A^*}}{k-1} - \frac{\alpha_{A^*}}{k} = \frac{\alpha_{A^*}}{k(k-1)} = \frac{\frac{\alpha_{A^*}}{k}\frac{\alpha_{A^*}}{k}}{\frac{\alpha_{A^*}}{k}(k-1)} = \frac{c_{xz}c_{yx}}{\sum_{w \in A} c_{wx}}.$$

Thus for the jump rate of reduced system defined as (34) holds the same relations of network reduction in the proof of Aldous' conjecture. Therefore we can use proposition 2.2.1 to say that when we reduce the system in a simple vertex the Spectral gap does not decreases. Then we have

$$\lambda_1^{RW}(G(\alpha^{(x_1)(x_2)\cdots(x_N)})) \geq \lambda_1^{RW}(G(\alpha^{(x_1)(x_2)\cdots(x_{N-1})})) \geq \cdots \geq \lambda_1^{RW}(G(\alpha)).$$

Thus $\lambda_1^{RW}(G(\alpha)) = \lambda_1(\alpha)$. ■

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