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Clifford regular domains

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One of the aims of multiplicative ideal theory is the description of an integral domain by means of the multiplicative semigroup of fractional ideals. In this context, the ideal class group $C(R)$, built by the isomorphism classes of invertible fractional ideals, has been one of the major objects of investigations. The ideal class group of the ring of integers of an algebraic number field was the first being studied in details. Afterwards several important generalizations of that particular case have been obtained in Commutative Algebra. Among them, the ideal class groups associated with star operations and ideal systems are the most general and fruitful ones (see [3],[15] and [25, Ch. 12]). In particular, the divisor class groups of Krull domains and Krull monoids are special cases of the previous concepts (for more details we refer to [23]).

Only recently the ideal class semigroup $S(R)$, built by the isomorphism classes of all nonzero fractional ideals, has been introduced and investigated by several authors. For instance, E.C. Dade, O. Taussky and H. Zassenhaus [19] investigated the structure of the ideal class semigroup of a non-principal order in algebraic number field. More generally, Halter-Koch [27] considered the case of the ideal class semigroup of lattices over Dedekind domains.

We focus on the interesting situation when the ideal class semigroup $S(R)$ is a Clifford semigroup.

Recall that a commutative semigroup S is a Clifford semigroup if every element a of S is regular (in the sense of von Neumann), i.e. $a = a^2x$ for some $x \in S$. It is known that a commutative semigroup S is a Clifford semigroup if and only if it is a disjoint union of groups H_e, where e ranges over the idempotent elements of S. Here H_e denotes the largest subgroup of S with identity element e and it is called the *constituent group* associated to e:

$$H_e = \{ae \mid abe = e \text{ for some } b \in S\}.$$

We say that a domain R is a *Clifford regular domain* if the ideal class semigroup $S(R)$ of R is a Clifford semigroup. To understand the structure of the ideal class semigroup of a Clifford regular domain R, one has to describe the idempotent elements of $S(R)$, the constituent group associated to them, and the bonding homomorphisms between those groups.

The first significant examples of Clifford regular domains are valuation domains. In fact, in [7], Bazzoni and Salce proved that the ideal class semigroup of any valuation domain is a Clifford semigroup whose constituent groups are either trivial or groups associated to the idempotent prime ideals P of R. The constituent groups are described as quotients of the form $\overline{\Gamma}/\Gamma$, where Γ is the value group of the localization R_P and $\overline{\Gamma}$ is the completion of Γ in the order topology; they are also called the *archimedean* groups of the localization R_P.

In 1994, Zanardo and Zannier studied the class semigroup of particular orders in algebraic number rings, proving that all orders in a quadratic field have Clifford class semigroup, whereas the ring of all entire function in the complex plane (which is Bezout) fails to have this property [45].

In [8] Bazzoni considered the case of a Prüfer domain proving that, if a Prüfer domain has finite character, i.e. every nonzero element is contained only in a finite number of maximal ideals, then it is Clifford regular. Later, in [9] and [10], for a Prüfer domains R of finite character, Bazzoni gave a description both of the idempotent elements of $S(R)$, and of their associated groups. The idempotents are shown to be either the isomorphism classes of fractional overrings having as associated group the ideal class groups of the overrings, or the isomorphism classes of products $P_1 \cdot P_2 \cdots P_n D$, where the P_i's are idempotent prime ideals of R, and D is a fractional overring of R. In this latter case, the associated group is an extension of the direct product of the archimedean groups of the localizations R_{P_i} by means of the ideal class group of D.

A complete characterization of the class of integrally closed Clifford regular domains was achieved in [11], where it was proved that this class coincides with the class of the Prüfer domains of finite character.

In addition [11] explores the relation between Clifford regularity, stability and finite stability. Recall that an ideal I of a domain R is said to be *stable* if it is invertible in its endomorphism ring, i.e. if $(I : I) = I((I : I) : I)$. A domain R is said to be *stable* if every ideal of R is stable. The notion of stability was first introduced in the Noetherian case with various different definitions which turned out to be equivalent in the case of a Noetherian local domain (for more details we refer to [36]). Olberding has described the structural properties of an arbitrary stable domain. In [35] and [36] he proved that a domain is stable if and only if it is of finite character and locally stable. Rush, in [39], considered the class of *finitely stable* domains, that is, domains with the property that every finitely generated ideal is stable and proved that the integral closure of such a domain is a Prüfer domain.

In [11] it is shown that the class of Clifford regular domains is properly intermediate between the class of stable domains and the class of finitely stable domains. In particular, from this, it follows that the integral closure of a Clifford regular domain is a Prüfer domain. In addition, we have that a Noetherian domain is Clifford regular if and only if it is a stable domain.

In [11] it was also outlined a relation between Clifford regularity and the *local invertibility property*. A domain R is said to have the local invertibility property if the following condition holds:

an ideal I of R is invertible if and only if every localization I_M at maximal ideal M of R is invertible, or equivalently principal.

In [8] and again in [11] the problem of determining whether, for a Prüfer domain, the local invertibility property is equivalent to the finite character condition was posed as a conjecture by Bazzoni, and so it is called the *Bazzoni's conjecture*. The question attracted the interest of many authors. Recently the validity of the Bazzoni's conjecture has been proved by Holland, Martinez, McGovern and Tesemma in [28], and independently by Halter-Koch in [25].

On the other hand it is possible to prove that if a Clifford regular domain R is either integrally closed or Noetherian, then it is of finite character (see the discussion at page 71 for the Noetherian case and Theorem 5.4.10 for integrally closed case). In the general case, the problem of determining whether Clifford regularity always implies finite character has been open for more than a decade. Very recently Bazzoni, in a still unpublished work, gave a positive answer to the problem ([12, Theorem 4.6]).

In this expository work we study the most important properties of Clifford regular domains. In particular in the Noetherian and Prüfer case.

Furthermore, following [26], we give a proof of Bazzoni's conjecture. Precisely, we show that a Prüfer domain R is Clifford regular if and only if R has finite character, if and only if the locally invertibility property holds in R.

We also give some conditions under which the Clifford regularity is inherited by overrings, i.e. if R is Clifford regular then every overring of R is also Clifford regular.

In Chapter 1, we define Clifford semigroups, and study various properties of regular and idempotent elements. Thanks to Clifford's theorem (Theorem 1.1.20) stating that if a semigroup is a union of groups then such union is disjoint, we can give a characterization of commutative Clifford semigroups.

Theorem 1 (Theorem 1.3.16)**.** *Let S be a commutative semigroup. Then S is Clifford if and only if it is a disjoint union of commutative groups.*

Moreover, $S = \bigcup_e H_e$ where e ranges over the set of all idempotent elements of S and $H_e := \{a \in S \mid ae = a \text{ and } ax = e \text{ for some } x \in S\}$ is the largest subgroup of S having e as identity element.

In Chapter 2, we introduce fractional ideals and invertible fractional ideals of a domain R and we study some of their properties. Recall that, given an integral domain R, an R-submodule I of $K = qf(R)$ is said to be a *fractional ideal* if there exists a nonzero element x of R such that $xI \subseteq R$.

Proposition 2 (Proposition 2.1.2)**.** *Let I be an R-submodule of K, then : I is a fractional ideal of R if and only if $I = \frac{1}{d}J$, where J is an ideal of R and $d \in R^*$.*

A fractional ideal I is said to be an *invertible fractional ideal* if there exists a fractional ideal J of R such that $IJ = JI = R$.

Lemma 3 (Lemma 2.2.3)**.** *Let I be a invertible fractional ideal, then there is a unique $J \in F(R)$ such that $IJ = JI = R$. Moreover, $J = (R : I)$.*

Finally, in Section 3 we introduce Prüfer domains, characterized by the property to be locally valuation domains (Theorem 2.3.8). Recall that, a domain R is called a *Bezout (respectively Prüfer) domain* if every nonzero finitely generated fractional ideal of R is principal (respectively invertible). An integral domain R is called a *valuation domain* if for every nonzero element $x \in K$ then either $x \in R$ or $x^{-1} \in R$.

Moreover, we show that if R is a Prüfer domain, then every overring T of R is still a Prüfer domain and can be represented as an intersection of localizations with respect to a particular set of prime ideals of R, that is $T = \bigcap\limits_{P \in S} R_P$ where $S = \{M \cap R \mid M \in Max(T)\}$ (Corollary 2.3.10).

In Chapter 3, we define the ideal class (semi)group of an integral domain. First, we prove that the ideal class semigroup $S(R)$ is isomorphic to factor semigroup $F(R)/P(R)$, where $F(R)$ denote the set of nonzero fractional ideals of R, and $P(R)$ denote the set of nonzero principal fractional ideals of R.

In Section 2 we study the regular elements of $S(R)$. The most important results of this section are:

Proposition 4 (Proposition 3.2.2)**.** *Let I be a fractional ideal of R. Then the following are equivalent:*

1. *$[I]$ is a regular element of $S(R)$.*
2. *$I = I^2X$ for some $X \in F(R)$.*
3. *$I = I^2(I : I^2)$.*
4. *I is a regular element of $F(R)$.*

Proposition 5 (Proposition 3.2.7)**.** *Let I be a nonzero ideal of R. Then the following hold:*

1. *If I is invertible in $E = (I : I)$, then $[I]$ is a regular element of $S(R)$.*
2. *If I is finitely generated, then I is invertible in $E = (I : I)$ if and only if $[I]$ is a regular element of $S(R)$.*

In Chapter 4 we define Clifford regular and (finitely) stable domains, and we examine the relations between these two classes of domains. In particular, we show that the class of Clifford regular domains stands between the class of stable domains and the class of finitely stable domains.

We also study the overrings of Clifford regular domains by giving some conditions for the Clifford regularity to be an ascent property. To do this, we use the surjectivity of the canonical map $\phi_T^R : R \to T$, $I \mapsto IT$ where T is an overring of R. In fact, let R be a Clifford regular domain and T be an overring of R. Suppose that the canonical map is surjective, i.e. every fractional ideals of T is extended from R, then T is a also a Clifford regular domain (Proposition 4.4.1). So, we can show the following proposition:

Proposition 6 (Proposition 4.4.2)**.** *If R is a Clifford regular domain and T is an overring of R, then T is Clifford regular in either of the following cases:*

1. *T is a fractional overring of R.*

2. *T is a localization of R.*

3. *T is a flat overring of R.*

4. *T is well-centered on R.*

5. *T is Noetherian.*

Furthermore, we prove that:

Proposition 7 (Proposition 4.4.7)**.** *Let R be an integrally closed Clifford regular domain, then every overring T of R is Clifford regular.*

In Chapter 5, we completely characterize integrally closed and Noetherian Clifford regular domains (Theorem 5.1.1 and Theorem 5.4.10).

Theorem 8 (Theorem 5.1.1)**.** *A Noetherian domain is Clifford regular if and only if it is stable.*

Moreover, we outline the interesting relation between Clifford regularity and local invertibility property. Thanks to the special properties of ideals in a Prüfer domain pointed out by Halter-Koch in [26], we give a proof of Bazzoni's conjecture.

Theorem 9 (Theorem 5.4.9)**.** *Let R be a Prüfer domain. Then the following conditions are equivalent:*

1. *R is Clifford regular.*

2. *R is of finite character.*

3. *R has the local invertibility property.*

Thus, since an integrally closed Clifford regular domain is a Prüfer domain (Proposition 4.1.10), the validity of Bazzoni's conjecture allows us to give the following characterization:

Theorem 10 (Theorem 5.4.10). *Let R be an integrally closed domain. Then, R is Clifford regular if and only if R is a Prüfer domain of finite character.*

In Chapter 6 we outline some recent results on more general classes of domains. We define star operations of an integral domain and we introduce star regularity and star stability.

The $\star$-stability was studied by Gabelli and Picozza in [21] for any star operation $\star$ and in particular when $\star = w$.

Kabbaj and Mimouni studied Boole and Clifford $\star$-regularity for $\star = t$ [29],[31],[32],[33]. In particular they extended the characterization of Prüfer Clifford regular domains to PvMDs.

Theorem 11 (Theorem 6.0.13). *A PvMD is Clifford t-regular if and only if it is a Krull-type domain (i.e. in a PvMD, Clifford t-regularity coincides with the finite t-character condition).*

Halter-Koch extended the results of Kabbaj and Mimouni to every star operation in the following sense ([26, Proposition 6.11] and [26, Proposition 6.12]):

Theorem 12 (Theorem 6.0.14). *Let R be a domain and let $\star$ be a star operation. Then the following hold:*

1. *R is Clifford $\star$-regular if and only if it is a Krull-type domain.*

2. *If R is $\star$-integrally closed and Clifford $\star$-regular, then R is a P$\star$MD and even a Krull-type domain.*

Finally the relations between Clifford star regularity and star stability was investigated in [22].

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