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Sintesi

Moser-Trudinger inequality and applications to a geometric problem

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Sintesi

In 1967 Neil Trudinger proved the following theorem

Theorem 0 (Trudinger inequality). *If $D = \{x \in \mathbb{R}^n \ : \ |x| < 1\}$ then there exists a constant $\alpha > 0$ such that*

$$\sup_{u \in W_0^{1,n}(D),\ \|\nabla u\|_n \leq 1} \int_D e^{\alpha |u|^{\frac{n}{n-1}}} dx < \infty \tag{1}$$

This result shows that the Sobolev embedding in the limiting case $p = n$ is of exponential type. It also shows that a function $u \in W_0^{1,n}(\Omega)$ has only singularities with logaritmic growth. Trudinger′ s proof makes use of the power expansion of $e^{\alpha u^2}$ and of Sobolev's precise estimates for the single terms of the expansion. However, with this approach, he was not able to identify the sharp value of α for which (1) holds that is

$$\alpha_n := \sup \left\{ \alpha > 0 \ : \sup_{u \in W_0^{1,n}(D)\ \|\nabla u\|_n \leq 1} \int_D e^{\alpha u^{\frac{n}{n-1}}} dx < \infty \right\}$$

Trudinger's result was later improved by Moser [8] who showed that $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$. He also showed that (1) holds on every bounded domain and that the sharp constant α_n does not depend on the domain.

Theorem 1 (Moser-Trudinger Inequality). *Let Ω be an open bounded domain in $\mathbb{R}^n$*

1. *There exists $C > 0$ such that $\forall 0 < \alpha \leq \alpha_n$* $\displaystyle\sup_{u \in W_0^{1,n}(\Omega),\ \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx \leq C|\Omega|.$

2. *If $\alpha > \alpha_n$ then* $\displaystyle\sup_{u \in W_0^{1,n}(\Omega),\ \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx = \infty.$

Over the years several extensions of Moser's theorem have been stated in different contexts. In particular we have studied Moser-Trudinger type inequalities on unbounded domain. Our starting point was a result by Mancini and Sandeep about Moser-Trudinger inequality on conformal disks. In [7] they considered the unit disk of $\mathbb{R}^2$ endowed with a Riemannian metric g conformally equivalent to the Euclidean metric g_e (i.e. $g = \lambda g_e$ for some positive smooth function λ). They obtained:

Theorem 2. *Let g be a Riemannian metric on D conformally equivalent to the Euclidean one then the following conditions are equivalent:*

1. *There exists* $C > 0$ *such that* $g \leq C g_h$ *where* $g_h = \frac{4}{(1-|x|^2)^2} g_e$*;*
2. $$\sup_{u \in \mathcal{H}_0^1(D), \|\nabla u\|_2 \leq 1} \int_D \left(e^{4\pi u^2} - 1 \right) dv_g < \infty$$
3. *There exists* $\alpha > 0$ *such that*
$$\sup_{u \in \mathcal{H}_0^1(D), \|\nabla u\|_2 \leq 1} \int_D \left(e^{\alpha u^2} - 1 \right) dv_g < \infty$$

Note that here the correction -1 is necessary since (D, g) could have infinite measure. We give a little improvement of this theorem showing that the conditions of the above theorem are equivalent to Poincaré's inequality on (D, g). More precisely denoting

$$\lambda_1(g) := \inf_{u \in H_0^1(D), u \neq 0} \frac{\int_D |\nabla u|^2 dx}{\int_\Omega u^2 dv_g} = \inf_{u \in H_0^1(D), u \neq 0} \frac{\int_D |\nabla u|^2 dx}{\int_\Omega u^2 \rho dx}$$

we have proved

Theorem 3. *Suppose that g is a Riemannian metric on D conformally equivalent to the Euclidean one then*
$$\lambda_1(g) > 0 \iff g \leq C g_h$$

In [7] it is shown that theorem 2 implies Moser-Trudinger inequality on simply connected domains of $\mathbb{R}^2$

Theorem 4. *Let* Ω *be a simply connected open subset of* $\mathbb{R}^2$*, then the following conditions are equivalent*

1. $\lambda_1(\Omega) > 0$*;*
2. $$\sup_{u \in \mathcal{H}_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega \left(e^{4\pi u^2} - 1 \right) dx < \infty$$
3. *there exists* $\alpha > 0$ *such that*
$$\sup_{u \in \mathcal{H}_0^1(\Omega), \|\nabla u\|_2 \leq 1} \int_\Omega \left(e^{\alpha u^2} - 1 \right) dx < \infty$$

Hence on simply connected domains Moser-Trudinder inequality is equivalent to Poincaré's. Mancini and Sandeep proposed the following question: Is Moser-Trudinger inequality equivalent to Poincaré's inequality on every open subset of $\mathbb{R}^2$?
We give an affermative answer to this question and we have proved that a similar equivalence holds also in higher dimension.
More precisely for any open subsets of $\mathbb{R}^n$ we have considered

$$\lambda(\Omega, n) := \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\left(\int_\Omega |\nabla u|^n dx\right)^{\frac{1}{n-1}}}{\int_\Omega |u|^{\frac{n}{n-1}}}$$

and we have proved that

Theorem 5. *Let Ω be an open subset of $\mathbb{R}^n$ then*

$$\sup_{u \in W_0^{1,n}(\Omega), ||\nabla u||_n \leq 1} \int_\Omega \left(e^{\alpha_n |u|^{\frac{n}{n-1}}} - 1\right) dx < \infty \Longleftrightarrow \lambda(\Omega, n) > 0$$

When $n = 2$ $\lambda(\Omega, n) = \lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ that is the best constant in Poincaré's inequality so that the condition $\lambda(\Omega, n) > 0$ is equivalent to Poincaré's inequality on Ω.
When $n \geq 3$ we do not know which are the domains for which $\lambda(\Omega, n) > 0$. It is even possible that this condition is never verified.

A Moser-Trudinger inequality on $\mathbb{R}^n$ was proved by Adachi and Tanaka in [1] removing from the Moser functional some terms of its Taylor expansion

Theorem 6. *If*

$$\Phi(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}$$

then $\forall\ \alpha < \alpha_n\ \exists\ C = C(\alpha, n)$ such that $\forall\ u \in W_0^{1,n}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \Phi\left(\alpha_n \left(\frac{|u|}{||\nabla u||_n}\right)^{\frac{n}{n-1}}\right) dx \leq C \frac{||u||_n^n}{||\nabla u||_n^n}$$

Moreover the inequality is false for $\alpha = \alpha_n$.

Bearing in mind this result we have proposed an extension of theorem 5. We have considered the function

$$\Phi_k(t) = e^t - \sum_{j=0}^{k-1} \frac{t^j}{j!}$$

and for $\Omega \subseteq \mathbb{R}^n$ the functional

$$F^k_\Omega(u) = \int_\Omega \Phi_k(\alpha_n u^{\frac{n}{n-1}})dx$$

which we call Moser's functional of order k. We have introduced the quantity

$$\lambda(\Omega, n, k) := \inf_{u \in W_0^{1,n}(\Omega), u \neq 0} \frac{\left(\int_\Omega |\nabla u|^n dx\right)^{\frac{k}{n-1}}}{\int_\Omega u^{\frac{nk}{n-1}} dx}$$

and extending naturally the procedure of the proof of theorem 5 we have proved

Theorem 7. *Let Ω be an open subset of $\mathbb{R}^n$ then*

$$\sup_{u \in W_0^{1,n}(\Omega), \|\nabla u\|_n \leq 1} F^k_\Omega(u) < \infty \Longleftrightarrow \lambda(\Omega, n, k) > 0$$

The result is particularly interesting when $k = n-1$. Indeed in this case the condition $\lambda(\Omega, n, n-1) > 0$ is equivalent to Poincaré's inequality on Ω. In the general case this shows that the Moser functional of order k on Ω is bounded if and only if there is a uniform control on the first term of its Taylor expansion.
We have also investigated whether it is possible to remove the hypothesis $\lambda(\Omega, n, k) > 0$ working on subspaces in which the first term of the expansion is bounded. We have obtained that this happens only for subcritical exponents.

In addition to the study of Moser's inequality on unbounded domain we have addressed the problem of the existence of extremal functions. First we have described the main results concernig the existence of extremal functions. These results are all based on the following concentration compactness theorem

Theorem 8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $u_k \in \mathcal{H}_\Omega$, $|\nabla u_k| dx \overset{k\to\infty}{\rightharpoonup} \mu$ and $u_k \overset{k\to\infty}{\rightharpoonup} u$ in $W_0^{1,n}(\Omega)$ then there exists a subsequence for which one of the following holds:*

1. *$u = 0$ and $\mu = \delta_{x_0}$ for some $x_0 \in \overline{\Omega}$.*

2. *$e^{\alpha_n |u_k|^{\frac{n}{n-1}}}$ is bounded in $L^p(\Omega)$ for some $p > 1$ and in particular $e^{\alpha_n |u_k|^{\frac{n}{n-1}}} \overset{k\to\infty}{\longrightarrow} e^{\alpha_n |u|^{\frac{n}{n-1}}}$ in $L^1(\Omega)$*

This state that if the supremum of Moser's functional is not attained then every maximizing sequence concentrates somewhere (i. e. $|\nabla u_k|^2 dv \rightharpoonup \delta_{x_0}$or some $x_0 \in \overline{\Omega}$). In particular if it is possible to exclude concentration for a maximizing sequence then the supremum is attained. This procedure was successfully used for the first time by Carleson-Chang in [2] for the n-dimensional unit ball where the use of symmetric rearrangement allows to consider only radially symmetric functions. They considered

$$C_{rad} = \sup\left\{\limsup_{k\to\infty}\int_\Omega e^{\alpha_n u_k^{\frac{n}{n-1}}}dx \;\middle|\; \{u_k\}\subseteq \mathcal{H}_D\cap W^{1,n}_{0,rad}(D), u_k \text{ concentrates at } 0\right\}$$

and proved

Theorem 9.

$$C_{rad} \leq \left(\frac{\omega_{n-1}}{n}\right)\left(1+e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}\right) < \sup_{u\in W^{1,n}_0(D),\ \|\nabla u\|_n\leq 1}\int_D e^{\alpha|u|^{\frac{n}{n-1}}}dx$$

and in particular the supremum is attained.

This result was later extended by Flucher who introduced the concentration function of Ω

$$C_\Omega(x) = \sup\left\{\limsup_{k\to\infty} F_\Omega(u_k) \mid u_k\in\mathcal{H}_\Omega \text{ concentrates at } x\right\}$$

and proved

Theorem 10. *If Ω is a bounded domain in $\mathbb{R}^n$ then*

$$\sup_{\overline{\Omega}} C_\Omega < \sup_{u\in W^{1,n}_0(\Omega),\ \|\nabla u\|_n\leq 1}\int_\Omega e^{4\pi u^2}dx$$

and in particular the supremum in the right side is attained

If Ω is unbounded then the concentration compactness theorem fails. Indeed a maximizing sequence for Moser's functional may also vanish at infinity. Hence to prove the existence of a maximizing sequence it is not sufficient to exclude concentration. In collaboration with Luca Battaglia we have proved an existence result for the two dimensional strip $\Omega = \mathbb{R}\times[-1,1]$. We have used Steiner's symmetrization to produce a maximizing sequence composed by functions which are symmetric and nonincreasing in both directions. Under this symmetry hypothesis, we were able to exclude concentration and to estimate the vanishing level with $\frac{4\pi}{\lambda_1(\Omega)}$,proving so that

Theorem 11. *For* $\Omega = \mathbb{R} \times [-1, 1]$ *if*

$$\sup_{u \in H_0^1(\Omega),\ \|\nabla u\|_2 \le 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} dx > \frac{4\pi}{\lambda_1(\Omega)} = \frac{16}{\pi}$$

then the supremum is attained.

Then we have produced a function for which the the Moser's functional is greater than the critical level $\frac{4\pi}{\lambda_1(\Omega)}$ proving so that

Theorem 12. *For* $\Omega = \mathbb{R} \times [-1, 1]$ $\sup_{u \in H_0^1(\Omega),\ \|\nabla u\|_2 \le 1} \int_\Omega e^{4\pi u^2} dx$ *is attained.*

To conclude the part of our work which is devoted to Moser-Trudinger inequality we have described the main results concerning Moser's inequality on compact Riemannian manifolds. In [8] it is shown that Moser-Trudinger inequality with the sharp exponent 4π holds on S^2 for functions with zero mean. This result was later extended to every compact Riemannian manifold. The subcritical inequality was proved by Cherrier in [4]

Theorem 13. *If* $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ *and* $\mathcal{H}(M, g) = \left\{ u \in W^{1,n}(M) \ : \ \|\nabla u\|_n \le 1, \ \int_M u = 0 \right\}$ *then*

$$\sup_{\mathcal{H}(M,g)} \int_M e^{\alpha |u|^{\frac{n}{n-1}}} dv_g < +\infty \qquad \forall \alpha < \alpha_n$$

$$\sup_{\mathcal{H}(M,g)} \int_M e^{\alpha |u|^{\frac{n}{n-1}}} dv_g = +\infty \qquad \forall \alpha > \alpha_n$$

while the sharp inequality was proved by Li in [6] where also the existence of extremal functions is proved.

Theorem 14. *Let* (M, g) *be an n-dimensional compact Riemannian manifold without boundary then*

1. $\forall\ \alpha \le \alpha_n$ $$\sup_{u \in W_0^{1,n}(M),\ \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha u^{\frac{n}{n-1}}} dx < \infty$$

   and the supremum is attained.

2. *for* $\alpha > \alpha_n$ $$\sup_{u \in W_0^{1,n}(M),\ \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha u^{\frac{n}{n-1}}} dx = +\infty$$

In the second part of our work we have described problems concerning the problem of prescribing the Gaussian curvature of a smooth manifold M in the conformal class of an assigned Riemannian metric of M. Given a two dimensional Riemannian manifold (M, g) and a smooth function K on M we wonder whether there exists a Riemannian metric $\tilde{g}$ conformally equivalent to g whose Gaussian curvature is K. A simple computation in normal coordinates shows that the Gaussian curvature of the metric $\tilde{g} = e^{2u}g$ is determined by the equation

$$\Delta_g u + Ke^{2u} = k_g \tag{2}$$

where k_g is the Gaussian curvature of (M, g) and Δ_g is the Laplace-Beltrami operator on (M, g).
If v is a smooth solution $\Delta_g v = k_g - \overline{k}_g$ it is simple to verify that u solves equation (2) if and only if $w = u - v$ solves

$$\Delta_g w + he^{2w} = \overline{k}_g \tag{3}$$

where $h = Ke^{2v}$. By the Gauss-Bonnet theorem $\overline{k}_g$ has the same sign as $\chi(M)$ so that the nature of the equation depends on the sign of $\chi(M)$.
In particular a direct integration of (3) gives that

Proposition 1. *A necessary condition for the solvability of (3) is*

1. *if* $\chi(M) < 0$ $\min_h < 0$.
2. *if* $\chi(M) = 0$ *or* $h \equiv 0$ *or* h *changes sign.*
3. *if* $\chi(M) < 0$ $\max_M h > 0$.

When $\chi(M) = 0$ solutions of (3) can be found searching critical points of the functional

$$J(u) = \int_M |\nabla u|^2 dv_g$$

on

$$B = \left\{ u \in H^1(M) \mid \int_M u = 0, \int_M he^{2u} dv_{\tilde{g}} = 0 \right\}$$

An application of Moser's inequality allows to prove that B is weakly closed so that since J is coercive and lower semicontinuous in the weak topology of $H^1(M)$ it is simple to find a minimum point of J. Computing the Euler-Lagrange equation associated to J it is simple to verify that if $\int_M h dv_g < 0$ then adding a suitable constant to the minimum point we find a solution of (3). It is also possible to verify by an integration of the equation that $\int_M h dv_g < 0$ is also a necessary condition for the solvability of (3). Hence

Theorem 15. *Let (M, g) be a two-dimensional Riemannian manifold with $\chi(M) = 0$ and let v be a solution of $\Delta_g v = k_g$. If $\tilde{g} = e^{2v} g$ then equation* (2) *has a solution if and only if $K \equiv 0$ or K changes sign and $\int_M K dv_{\tilde{g}} < 0$.*

When $\chi(M) < 0$ some existence results can be found with the method of upper and lower solutions. In [5] the following existence result is proved:

Proposition 2. *Let (M, g) be a Riemannian manifold with $\chi(M) < 0$ and let h be a smooth function on M with $\overline{h} < 0$ then there exists a constant $-\infty \leq c = c(h) < 0$ such that equation (3) has a solution if $c < \overline{k_g}$ and has no solution if $c > \overline{k_g}$.*

If h is nonpositive then the problem is more simple and if $\min_M h < 0$ it is often possible to find a solution

Theorem 16. *Let (M, g) be a two dimensional compact Riemannian manifold with $\chi(M) < 0$ and let $K \in C^\infty(M)$, be a nonpositive function then equation (2) has a solution if and only if $K \not\equiv 0$.*

When $\chi(M) > 0$ solutions can be found adding a suitable constant to the critical points of

$$F_h(u) = \int_M |\nabla_g u|^2 dv_g + 2\overline{k}_g \int_M u \, dv_g - 2\pi\chi(M) \log\left(\frac{1}{2\pi\chi(M)} \int_M he^{2u}\right)$$

on the open subset of $H^1(S^2)$

$$O_h := \left\{ u \in H^1(M) \ : \ \int_M he^{2u} dv_g > 0 \right\}$$

As a consequence of Moser's inequality on M it is simple to find that

Lemma 1. *There exists a constant $C > 0$ such that $\forall \ a \in \mathbb{R}$*

$$\int_M e^{au} dv_g \leq Ce^{\frac{a^2}{16\pi} \int_M |\nabla_g u|_g^2 dv_g + a\overline{u}} \qquad \forall \ u \in H^1(M)$$

This lemma allows to prove

Proposition 3. *Let (M, g) be a compact Riemannian manifold with $\chi(M) > 0$ then $\forall h \in C^\infty(M)$ with $\max_h > 0$ there exists $C > 0$ such that*

$$F_h(u) = \left(1 - \frac{\chi(M)}{2}\right) \int_M |\nabla_g u|^2 dv_g - C$$

$\forall \ u \in O_h$.

When $\chi(M) = 1$ this proposition states that F_h is coercive on O_h and it is simple to find the existence of a solution.

Theorem 17. *Let (M, g) be a compact Riemannian manifold with $\chi(M) = 1$. If $K \in C^\infty(M)$ then equation (2) is solvable if and only if $\max_M K > 0$.*

When $\chi(M) = 2$ (essentially the case of S^2) proposition 3 only states that F_K (we remark that for the sphere (2),(3) are equal) is bounded from below. Hence in the case of S^2 the lack of coercivity makes the problem more difficoult. It is also possible to prove

Lemma 2. *If $K \in C^\infty(S^2)$ is a positive function and F_K has a minimum point on O_K then K is constant.*

Hence, except for the trivial case $K \equiv c$, a minimum point does not exist. To find solution there are two main ways:

- find minimum points for F_K on suitable subspaces of O_h.
- search for saddle points of F_K.

The first strategy was successfully introduced by Moser. He proved that for even functions the sharp exponent in Moser's inequality doubles.

Proposition 4 (Moser's inequality for even functions)**.** *There exists $C > 0$ such that for each even function $u \in H^1_{even}(S^2)$ with $\int_{S^2} |\nabla u|^2 dv_{g_0} \le 1$ and $\int_{S^2} u \, dv_{g_0} = 0$*

$$\int_{S^2} e^{8\pi u^2} dv_{g_0} \le C$$

This provides a sharp form of lemma 1 which allows to recover the coercivity of F_K. In particular it is possible to obtain

Theorem 18. *If $K \in C^\infty(S^2)$ is an even function then equation (2) is solvable if and only if $\max_M K > 0$.*

This result justifies Moser's interest in sharp exponents for Trudinger's inequality. Trying to repeat Moser's strategy several improvements of lemma 1 have been stated.

Lemma 3 (Aubin's inequality)**.** *If $u \in H^1(S^2)$ and $\int_{S^2} e^{2u} x_i dv_g = 0$ for $i = 1, 2, 3$ then*

$$\int_{S^2} e^{2u} dv_{g_0} \le C(\varepsilon) e^{\left(\frac{1}{8\pi} + \varepsilon\right) \int_M |\nabla u|^2 dv_g + 2\overline{u}}$$

Lemma 4 (Onofri's inequality).

$$\forall\ u \in H^1(S^2) \qquad \int_{S^2} e^{au} dv_g \leq e^{\frac{a^2}{16\pi}\int_{S^2}|\nabla u|^2 dv_{g_0} + \frac{a}{2\pi}\int_{S^2} u\ dv_g}$$

Lemma 5 (Chang-Yang inequality). *If $u \in H^1(S^2)$ and $\int_{S^2} e^{2u} x_i dv_g = 0$ for $i = 1,2,3$ then there exists $\frac{1}{2} < a < 1$*

$$\int_{S^2} e^{2u} dv_g \leq e^{\frac{a}{4\pi}\int_{S^2}|\nabla u|^2 dv_{g_0} + 2\overline{u}}$$

However these results have not a direct application to the resolution of (2) (contrary to Moser's one) so that they do not give remarkable improvements of Moser's theorem for even functions.
The second approach to the problem (the research of saddle points) was introduced by Chang and Yang in [3]. They were able to produce a suitable min-max scheme for the functional F_K. We describe briefly the main steps of their results. Given a function $u \in H^1(S^2)$ we define the center of mass of e^{2u} (denoted by $C.M.(e^{2u})$) the point in $\mathbb{R}^3$ of coordinates $\int_{S^2} x_j e^{2u} dv_{g_0}$ $j = 1,2,3$. Starting from two nondegenerated maximum points P_1, P_2 of a positive function K they considered a set $\mathcal{P}(P_1, P_2)$ of continuous paths $s \longrightarrow u_s$ in $H^1(S^2)$ such that $\lim_{s\to+\infty} C.M.(e^{2u}) = P_1$, $\lim_{s\to-\infty} C.M.(e^{2u}) = P_2$. Suitable assumptions on the paths allow to prove that $s \longrightarrow F(u_s)$ has a maximum point u_{s_k}. They considered $c = \inf_{\mathcal{P}} \max_s (u_s)$. Given a minimizing sequence of paths u_k they considered the maximum points u_k, s_k. If u_k, s_k is bounded in $H^1(S^2)$ then it converges weakly to a function u which turns out to be a solution of equation 2. Usine lemma 1 they proved the following concentration-compactness lemma:

Proposition 5 (Concetration lemma). *Let $u_j \in H^1(S^2)$ such that $\int_{S^2} e^{2u_j} = 4\pi$ and $\int_{S^2}|\nabla u_j|^2 + 2\int_{S^2} u_j \leq C$ then on a subsequence one of the following holds:*

1. $\exists\, C' > 0$ *such that* $\int_{S^2} |\nabla u_j|^2 d\mu \leq C'$
2. $\{e^{2u_j}\}$ *concentrates at a point* Q

In particular to prove the existence of solutions it is sufficient to exclude the concentration of u_{k,s_k}. Using sophisticated techniques they showed

Proposition 6. *If $e^{2u_{k,s_k}}$ concentrates at $Q \in S^2$ then replacing (if necessary) u_k with another minimizing sequence of paths we can suppose*

1. *Q is a critical point of K*
2. *Q is not a nondegenerated local maximum for K*
3. *Q is not a local minimum or a saddle point in which $\Delta K > 0$.*

In particular if K has only these types of critical points then concentration is excluded and (2) has a solution.

Theorem 19. *Let K be a smooth positive function with non degenerated critical points and in addiction $\Delta K(Q) \neq 0$ in all these critical points. If K has at least two loacal maxima and in all the saddle points $\Delta K > 0$ then equation (2) admists a solution.*

We remark that in the analysis of concentration all the sharp forms of lemma 1 are used. Hence lemma 1 (which is a consequence of Moser's inequality) has a key role in the problem.

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