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Resolution of singularities for plane curves

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The main argument of this thesis is the study of singularities of plane curves, and the role played by Cremonian transformations in their resolution.
The concept of sequence of blowing up is used as a key tool to achieve the resolution of singularities.

The topics of this thesis go back to the important work done by the so-called **Italian school of algebraic geometry** on the birational geometry (1885-1935). There were 30 or 40 mathematicians who played the major contributions, in fact, half of them were Italian. No doubt the leadership lies with the Rome group of Guido Castelnuovo (1865-1952), Federigo Enriques (1871-1946) and Francesco Severi (1879-1961), who were involved in some of the deepest discoveries under the impulse of the previous studies of Luigi Cremona (1830-1903) considered the founder of the Italian school of algebraic geometry.

In 1854 Luigi Cremona who was professor of mathematics in several Italian universities introduced the general birational transformations of the whole plane in itself and gave important works that were summarized and systematized by German mathematician Max Noether (1844-1921) and then by Castelnuovo. For example the important theorem that states that every Cremonian transformation is the composition of an automorphism and a finite number of elementary quadratic transformation; in this thesis is in fact called the Noether and Castelnuovo theorem.

Guido Castelnuovo took over Cremona's job when he later died in 1903. He took his chair of superior mathematics in Rome and collaborated with Federigo Enriques. This collaboration started in 1892 when Enriques was a young graduate, but grew further over the next 20 years so that he became an important member of the Italian school of algebraic geometry.
Together they did basic research on the theory of algebraic curves (the argu-

ment in the thesis) and their classification by genus, then Enriques focused on his famous study of the complete birational classification of algebraic surfaces, until he lost all his academic positions in 1938, when the fascist government enacted the racial laws, which in particular banned Jews from holding professorships in universities.

In the mid-thirties was born a movement of revision of algebraic geometry that had among its members André Weil (1906-1998) and Oscar Zariski(1899-1986) and aimed to reconstruct in a modern and foundational way the main results of Italian school, and that lead to the emergence of modern algebraic geometry.

Let us now describe in more detail the contents of this thesis.

In the first chapter we collect some peliminary results on what is needed from algebraic geometry.

Since we deal with plane curves we have maintained our preliminaries at the most basic level; it is almost enough to know the theory of projective varieties at the level of Shafarevich's book, or better, Fulton's beautiful book on algebraic curves.

Additionally some intersection theory is needed, but only for what concernes curves on rational surfaces.

In the second chapter we introduce the concept of blowing up and its main properties.

We give some simple examples of how it can be used in the solution of the singularities of the nodal cubic, the cusp and the tacnode.

Then we present in a modern notation the concepts of infinitely near, satellite and proximate points, which were already introduced in the Enriques' work and we define proximity matrix and the admissible oriented graphs which encode sequences of blowing up.

Through this tool we are allowed to write the fundamental formulas in a simple and algorithmic way.

To "resolve the singularities" of a projective curve C means to construct a non-singular projective curve X and a birational morphism $f : X \to C$. A rough idea of the procedure we will follow is this: If $C \subset \mathbb{P}^2$, and P is a multiple point on C, we will remove the point P from $\mathbb{P}^2$ and replace it by a projective line L. The points of L will correspond to the tangent directions at P. This can be done in such a way that the resulting "blown up" plane $B = (\mathbb{P}^2 - \{P\}) \cup L$ is still a variety, and, in fact, a variety covered by open sets isomorphic to $\mathbb{A}^2$. The curve C will be birationally equivalent to a curve C' on B, with $C' - (C' \cap L)$ isomorphic to $C - \{P\}$; but C' will have "better" multiple points on L than C has at P.

In this chapter we study birational transformation through the essential notion of *infinitesimal neighbourhood of a point*. This one shall be defined in a "geometric" way with the *Blowing up*

Definition 0.1. *Let T be a smooth surface. Blowing up a point P in T is still a smooth surface S furnished with a morphism $\sigma : S \to T$ such that:*

- *$E := \sigma^{-1}(P)$ is a rational curve, i.e. $E \cong \mathbb{P}^1$, smooth in S;*
- *$S \backslash E$ is isomorphic to $T \backslash \{P\}$.*

*S is determined up to isomorphism (i.e. essentially unique) by the two previous properties. From now on we say that S is obtained by T with P blown up. The curve E is said to be **the exceptional curve** of the blowing up.*

Definition 0.2. *(Strict transform, inverse image, and total transform)*
Let $\sigma : S \to T$ the blowing up of T in P as in definition 0.1 and let C be a reduced curve of T.
Then we define the strict transform $\tilde{C}$ of C in S as it follows:

$$\tilde{C} = \overline{\sigma^{-1}(C \backslash \{P\})}$$

where the bar denotes the topological closure in S
We say that $\sigma^{-1}(C)$ is the (total) inverse image of C.
In particular $\sigma^{-1}(C)$ coincides with $\tilde{C}$ if and only if $P \notin C$, else we find:

$$\sigma^{-1}(C) = \tilde{C} \cup E. \tag{1}$$

We define the total transform $\sigma^*(C)$ *of* C *in* S *as it follows:*

$$\sigma^*(C) = \tilde{C} + nE, \quad n = mult_C(P) \tag{2}$$

where n *is the multiplicity of* C *in* P*. We observe that* $\sigma^*(C)$ *is a divisor and its support is* $\sigma^{-1}(C)$*, i.e. the inverse image coincides in set theory with the total transform.*

We see some basic examples of resolution of singularity through the blowing up of a point:

1) **The nodal cubic**

Let $C \in \mathbb{A}^2$ be the cubic of equation $y^2 = x^2(x+1)$, that is the cubic of equation $y^2z = x^3 + x^2z$ in $\mathbb{P}^2$. This is an irreducible cubic curve with one node in the origin. Then blowing up the origin and looking in the chart $u \neq 0$, $\sigma^{-1}(C)$ is given by $\{y^2 = x^2(x+1),\ y = xv\}$ (we have set $u = 1$). Substituting we get $\{(a)\ x^2(v^2 - x - 1) = 0,\ (b)\ y = xv\}$. Equation (a) consists of two components in the coordinates (x, y, v) of $\mathbb{A}^3$. The first is given by $\{x^2 = 0,\ y = 0\}$ and is therefore that equation that allows us to say that E is counted with multiplicity two in $\sigma^{-1}(C)$. The other component is given by $\{v^2 - x - 1 = 0,\ y = xv\}$ and is the one we have denoted by $\tilde{C}$. This component intersects the exceptional divisor in the points given by $v^2 = 1$, i.e. $v = 1, -1$ that correspond to the principal tangents of C at O.

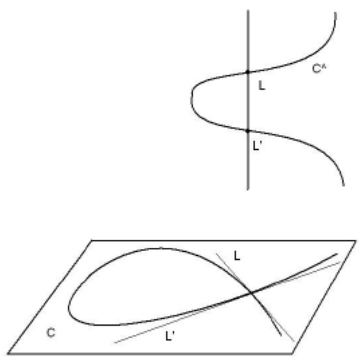

Figure 0.1: **The nodal cubic**

2) **The cusp**

Let $C \in \mathbb{P}^2$ be a cubic of equation $zy^2 - x^3 = 0$, i.e. the cubic of equation $y^2 - x^3 = 0$ in $\mathbb{A}^2$ by the local chart $z \neq 0$. We study the equation of blowing up and we have therefore, that the exceptional divisor is tangent to $\tilde{C}$ and the multiplicity of the intersection is two, but $\tilde{C}$ is a smooth curve. We can improve this situation through another blowing up.

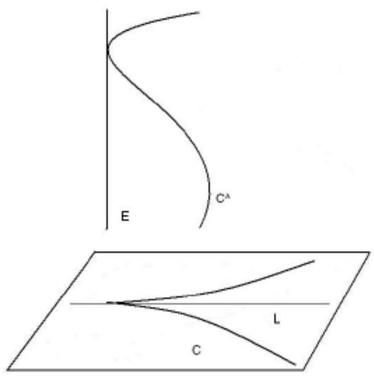

Figure 0.2**: The cusp**

3) **The tacnode**

Let $C \in \mathbb{P}^2$ be the curve of equation $zx^2 - x^4 - y^4 = 0$, i.e. the curve of equation $x^2 - x^4 - y^4 = 0$ in $\mathbb{A}^2$ by the local chart $z \neq 0$. We see that in this case, a single blowing up has not solved the singularity and the multiplicity is always two.

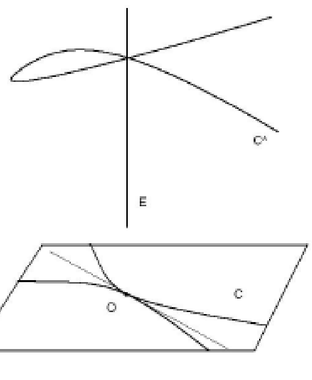

Figure 0.3**: The tacnode**

For this reason we define:

Definition 0.3. *(A sequence of blowing up) Let P_1 be a point on a (smooth) surface $T = S_0$. We consider the blowing up $\sigma_1 : S_1 \to T$ in P_1 and we indicate with $E_1^1 = \sigma_1^{-1}(P_1)$ the exceptional curve.*
Then let $P_2 \in S_1$ and $\sigma_2 : S_2 \to S_1$ be the blowing up of S_1 in P_2. We denote the exceptional curve by E_2^2 and the strict transform of E_1^1 in S_2 by E_1^2. We observe that if $P_2 \notin E_1^1$, then the total transform of E_1^1 in S_2 coincides with the strict transform E_1^2. Else, if $P_2 \in E_1^1$, by the formulas 1 and 2 it follows that:

$$(\sigma_2 \circ \sigma_1)^{-1}(P_1) = \sigma_2^{-1}(E_1^1) = E_1^2 \cup E_2^2 \quad \text{and} \quad \sigma_2^*(E_1^1) = E_1^2 + E_2^2$$

Repeating the construction r times, we define for all $i = 1, ...r$

- *the blowing up $\sigma_i : S_i \to S_{i-1}$ of S_{i-1} in P_i;*
- *the exceptional curve $E_i^i = \sigma_i^{-1}(P_i)$ of S_i;*
- *for all $j > i$, the composition $\sigma_{ji} : S_j \to S_{i-1} = \sigma_j \circ \sigma_{j-1} \circ ... \circ \sigma_i$;*
- *the total transform $E_i^* = \sigma_{r,i+1}^*(E_i^i)$ of P_i in $S = S_r$;*
- *for all $j > i$, the strict transform E_i^j of E_i^i in S_j;*
- *the strict transform $E_i := E_i^r$ of E_i^i in S;*
- *$(\ , \ )_i$ and $(\ , \ )$ respectively the intersection number in S_i and in S.*

All these data form the sequence of blowing up $\sigma = \sigma_{r1} : S \to T$ in the points $P_1, ..., P_r$. From now on, with abuse of notation, we say that E_i and E_i^* are respectively the strict and the total transform of points P_i in S
Sometimes we shall consider a sequence of blowing up from $i = 0$, i.e. with $T = S_{-1}$. We observe that the strict transforms E_i^j can be defined inductively:

$$E_i^j = \begin{cases} \sigma_j^*(E_i^{j-1}) & \text{if } P_j \notin E_i^{j-1} \\ \sigma_j^*(E_i^{j-1}) - E_j^j & \text{if } P_j \in E_i^{j-1} \end{cases} \tag{3}$$

for all $j > i$, as it follows by 2.

Let $\sigma_1 : S_1 \to T$ be the first blowing up, normally T will be the projective plane $\mathbb{P}^2$. the exceptional curve E_1^1 is called *first-order infinitesimal neighbourhood* of P_i and its point are called **proximate** to P_1. We suppose that the second point blown up is proximate to the first, i.e. $P_2 \in E_1^1$. A point of E_1^2 is said to be *proximate* to P_1, while a point of E_2^2 is said **infinitely near** of order 2 to P_1. We say that the point $P' = E_1^2 \cap E_2^2$ shown in figure 0.4, is **satellite** for P_1.

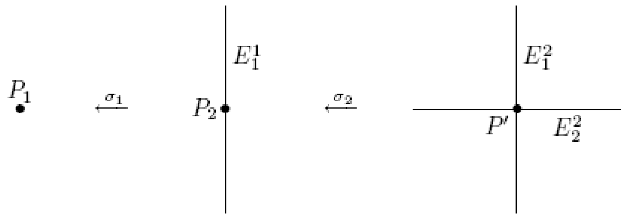

Figure 0.4**: Blowing up a point proximate to another**

In general a point $Q \in S_j$ is said to be **infinitely near** to $P_i =: P$ if it is $Q \in \sigma_{ji}^{-1}(P)$ and we write:

$$Q > P. \tag{4}$$

Moreover Q is said to be *proximate* to P if we also have $Q \in E_i^j$ and we write:

$$Q \to P. \tag{5}$$

Obviously a proximate point to P is also infinitely near to P. We define *infinitesimal order* of a point by induction on the number r of blown up points. The basis for the induction is $r = 0$. Points of T are said *proper*, or infinitely near of order 0, on T. Now let be $Q \in S_r$. We recall that for definition 0.1 there exists the isomorphism:

$$\sigma_{r|S_r \setminus E_r^r} : S_r \setminus E_r^r \xrightarrow{\sim} S_{r-1} \setminus \{P_r\}.$$

If $Q \notin E_r^r$, then we say that Q has the same *infinitesimal order* on T of $\sigma_r(Q) \in S_{r-1}$, that is known by inductive hipothesis. If instead $Q \in E_r^r$, then by inductive hipothesis $P_r \in S_{r-1}$ has a certain infinitesimal order s on T and we say that Q is *infinitely near* of order $s+1$, or that Q has *infinitesimal order* $s+1$ on T.

We consider a point $Q \in S_j$ infinitely near to $P_i = P \in S_{i-1}$, in symbols $Q > P$. If s is the infinitesimal order of Q on S_{i-1}, then we say that Q is infinitely near of *order* s to P and we write :

$$Q >^s P. \tag{6}$$

The set of points infinitely near of order s to a point P is called *infinitesimal neighbourhood of order* s of P. Naturally an infinitely near point of order 1 to P is also proximate to P. A point Q proximate to P but infinitely near of a higher order than 1 to P, for example the point P' in figure 0.4 is said to be **satellite** for P and we write:

$$Q \odot P.$$

So the symbolic definition of satellite is:

$$Q \odot P \Longleftrightarrow Q \rightarrow P \text{ } e \text{ } Q \not>^1 P.$$

An infinitely near point that is not satellite is said **free**.

We observe that, by definition of infinitesimal order, if $Q >^s P$ then there exist $s-1$ ponts P_{i_j} infinitely near to P, with $j = 1, ..., s-1$, such that:

$$Q >^1 P_{i_{s-1}} >^1 P_{i_{s-2}} >^1 ... >^1 P_{i_2} >^1 P_{i_1} >^1 P. \tag{7}$$

Conversely if there exists a succession of the kind of 7, then Q is infinitely near of order s to P.

The proper points, i.e. of order infinitesimal 0, on T correspond to the points belonging to T and so we write:

$$P \in T$$

if and only if P is a proper point on T. A point Q is said to be at a finite distance from a proper point P if $Q \not> P$. From now on we shall talk about *points on* a surface T, including both points proper and infinitely near ones. By definition a point on $T = \mathbb{P}^2$ is simply a point proper of a surface S obtained by T through a finite number of blowing up.

Also in this chapter, we continue to use notations introduced in def 0.3 to understand the geometry of surfaces obtained by the blowing up, which are rational if the starting surface is $\mathbb{P}^2$, we must define some combinatorial structure that records the way in which we have blown up. In particular we want to calculate an explicit formula for expressing the strict transforms E_i of the points blown up as a linear combination of their total transforms E_i^* (and vice versa). We have already seen with the formula 3 an inductive definition of E_i^j and then of E_i. Then we also have that to calculate the E_i in function of E_j^* we need to know only the relative positions of the blown up points.

We associate with a sequence of blowing up $\sigma : S \to T$ of r points a matrix $Q = (q_{ij})$ of order $(r \times r)$ defined by:

$$q_{ij} = \begin{cases} 1 & if\ P_j \in E_i^{j-1} \\ 0 & if\ P_j \notin E_i^{j-1}. \end{cases} \tag{8}$$

for $i < j$, that encodes exactly the relationship of proximity between the P_i:

$$q_{ij} = \begin{cases} 1 & if\ P_j \to P_i,\ i.e.\ if\ P_j\ is\ proximate\ to\ P_i \\ 0 & else \end{cases}$$

and which therefore we call a *proximity matrix* associated with the sequence of blowing up σ, or more simply a *proximity matrix* of σ.

We observe that two proximity matrices Q and Q' associated with the same sequence of blowing up are similar to each other by a permutation matrix, i.e, we find that:

$$Q' = A^{-1}QA,$$

for some permutation matrix A. Here are the basic properties of a proximity matrix Q:

1. Q is strictly upper triangular;
2. the entries of Q are only 0 or 1;
3. in each column there are at most two nonzero entries;
4. if $q_{ik} = q_{jk} = 1$ and $j > i$, then $q_{ij} = 1$;
5. no column with two nonzero entries is repeated.

Now we list other (trivial) properties of a proximity matrix Q associated with a sequence of blowing up σ:

- if $P_j >^1 P_i$, then $q_{ij} = 1$;
- the j-th column of Q is zero if and only if $P_j \in T$;
- if $E_i \cap E_j \neq \emptyset$ and $j > i$, then $q_{ij} = 1$;
- if $q_{ij} = 1$ and $E_i \cap E_j = \emptyset$, then there exists k such that $q_{ik} = q_{jk} = 1$;
- P_k is a satellite if and only if the k-th column of Q has two nonzero entries;
- if $q_{ik} = q_{jk} = 1$ with $j > i$, then $P_k \odot P_i$.

We recall that we find a satellite point only when we blow up the intersection (of the strict transforms) of two exceptional curves.

Lemma 0.1. *Let $\sigma : S \to T$ be a sequence of r blowing up, Then:*

$$Pic\, S \cong Pic\, T \oplus \mathbb{Z}^r,$$

where $Pic\, T \hookrightarrow Pic\, S$ is given by $C \mapsto \sigma^*(C)$ and $\{E_i\}_{1 \le i \le r}$ is a set of generators of $\mathbb{Z}^r$. Denoted by $Q = (q_{ij})$ a proximity matrix associated with σ, in $Pic\, S$ is the following formula:

$$E_j = E_j^* - \sum_{k \neq j} q_{jk} E_k^* = E_j^* - \sum_{k : P_k \to P_j} E_k^*. \tag{9}$$

In addition, the total transforms E_i^* of P_i have the following expression in $\operatorname{Pic} S$:

$$E_i^* = \sum_{j=1}^{r} m_{ij} E_j, \quad \text{with } M = I + Q + Q^2 + \cdots + Q^{r-1} = \sum_{k=0}^{r-1} Q^k \tag{10}$$

where I is the identity matrix and $M = (m_{ij})$ is an upper triangular matrix with integer coefficients. Finally, the intersections numbers between the E_i are:

$$(E_i, E_j) = \begin{cases} < 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \text{ and } E_i \cup E_j \neq \emptyset \\ 0 & \text{else} \end{cases} \tag{11}$$

and in particular we have:

$$(E_i, E_i) = -1 - \sharp \{k \mid P_k \to P_i\}, \tag{12}$$

in other words the opposite of the self-intersection of E_i is the number of blown up points proximate to P_i increased by 1.

Lemma 0.2. *A matrix Q of order $r \times r$ is associated to a sequence of blowing up of r points if and only if Q satisfies the properties of a proximity matrix.*

Remark 0.1. *Although the infinitely near points are a classical concept, the formulas calculated in this section are not explicitly written in the literature, in fact are implicitly applied several times in the book of Enriques [10]. The originality of our approach is the introduction of a proximity matrix Q associated with a sequence of blowing up, that allows us to handle formulas in an easier way. For example expressing the total transforms of the blown up points in function of their strict transforms.*

Then we will see how to interpret the proximity matrix as an oriented graph.

To express the strict transforms in function of the total transforms we started from the proximity matrix associated with a sequence of blowing up. To delete the dependence of the proximity matrix from the order of the blown up points, is convenient to encode the same data with an oriented graph.

A sequence of blowing up defines an oriented graph in a natural way: the vertices are the points blown up and the arrows are given by the relations of proximity, which in fact we have indicated with the same symbol, recall the notation 4, 5 of infinitelly near and proximate point. The properties of the proximity matrix of a sequence of blowing up are exactly the same as the adjacency matrix of an oriented admissible graph, according to a sort that makes the adjacency matrix strictly upper triangular. Then we associate a sequence of blowing up to an oriented admissible graph. Conversely, the adjacency matrix of an oriented admissible graph satisfies the properties of the proximity matrix and for the lemma 0.2 is the proximity matrix of a sequence of blowing up.

Theorem 0.1. *There is a one-to-one correspondence between oriented admissible graphs with r vertices and the equivalence classes of sequences of r blowing up.*

Example

We are blowing up 6 points in the following way:

$$P_1 \in T, \;\; P_2 >^1 P_1, \;\; P_3 = E_1^2 \cap E_2^2, \;\; P_4 >^1 P_3, \;\; P_5 = E_3^4 \cap E_4^4, \;\; P_6 = E_1^5 \cap E_3^5,$$

as shown in figure 0.5. Then P_1 is the only point proper and $P_2, ..., P_6$ are all infinitely near to P_1. In particular P_3, P_5 and P_6 are satellite, while P_2 and P_4 are free. We observe that we have $P_3 \odot P_1$ while $P_3 \not\odot P_2$, because $P_3 >^1 P_2$. Like wise we can see that $P_5 \odot P_3$ and $P_6 \odot P_1$, but $P_5 \not\odot P_4$ and $P_6 \not\odot P_3$. Finally, we note that $P_2,\ P_3,\ P_4$ and P_5 are infinitely near to P_1 of order respectively 1, 2, 3 e 4 , while $P_6 >^2 P_2$ because $P_6 >^1 P_3 >^1 P_1$.

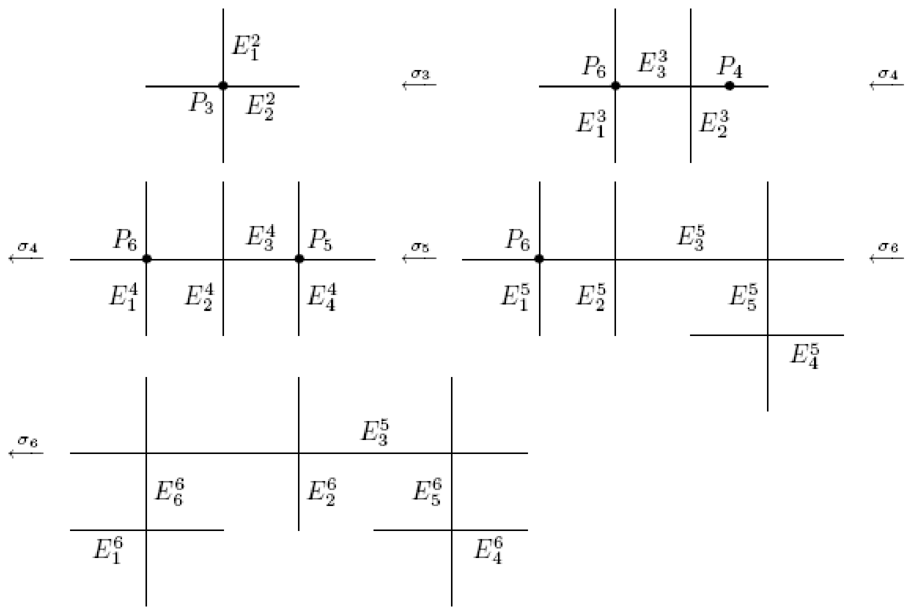

Figure 0.5**: Example of sequence of blowing up**

A proximity matrix Q associated to σ is:

$$Q = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 1 \\ \cdot & 0 & 1 & 1 & 0 & 0 \\ \cdot & & 0 & 1 & 1 & 1 \\ \cdot & & & 0 & 1 & 0 \\ \cdot & & & & 0 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$$

and applying the lemma 0.1 we calculate M and the intersection numbers between the E_i:

$$M = \begin{pmatrix} 1 & 1 & 2 & 2 & 4 & 3 \\ 0 & 1 & 1 & 1 & 2 & 1 \\ \cdot & 0 & 1 & 1 & 2 & 1 \\ \cdot & & 0 & 1 & 1 & 0 \\ \cdot & & & 0 & 1 & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & 1 \end{pmatrix} \qquad (E_i, E_j) = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

while $(E_i^*, E_j^*) = -I$, as always. The sequence of blowing up in the example determines the graph of figure, where the numbers associated with the vertices are indicated for convenience of the reader.

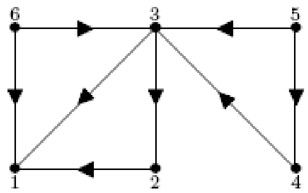

Figure 0.6**: Oriented graph of the blowing up sequence in the example**

In the third chapter we recall the results about the resolution of singularities of a curve on a surface. We show that oriented admissible graphs, weighted with the multiplicity of the singular points, uniquely determine the classes of equvalence of singularities, and they generalize the diagrams introduced by Enriques to study the Puiseux series expansion of the branches of the curves (we give a brief recall of his work at the end of this chapter). Moreover we present a procedure, called unloading principle, through which we can calculate effective multiplicities of a divisor from the virtual ones.

An irreducible curve has only a finite number of singularities, included $infinitely$ *near point*.

Theorem 0.2. *Let C be an irreducible curve in a smooth surface T. Then there exist a sequence of blowing up $\sigma : S \to T$ such that the strict transform of C in S is smooth.*

With the previous theorem we *solved the singularities* of C in T.

Remark 0.2. *Let C be a plane curve, i.e. $T = \mathbb{P}^2$. We have just seen how to obtain a desingularization $\tilde{C}$ of C, but the surface in which is $\tilde{C}$ is in general more complicated than $\mathbb{P}^2$. We shall study in chapter four another way to solve (partially) singularities of a plane curve staying in the plane, i.e. using the quadratic transformation, according to Noether's method.*

Then we give the *embedded resolution of singularities* of C in T

Theorem 0.3. *Let C be any curve of a smooth surface T. Then exists a sequence of blowing up $\sigma : S \to T$ such that the inverse image $\sigma^{-1}(C)$ in S has transverse intersection, i.e. its irriducible components are smooth and the only singularities are nodes.*

A problem, already addressed and solved by the classic, is to calculate the effective multiplicities of a divisor starting by the virtual ones.

Let D' be an effective divisor of S of degree d and (virtual) multiplicities a'_i in P_i. We compute the effective multiplicities of D' with the so called unloading principle in the following way :

1. We define $D := D'$ and $E := 0$.
2. We put $i := 1$.
3. If $(D, E_i) < 0$, the we redefine $D := D - E_i$ and $E := E + E_i$ and we go back to step 2
4. We increment i of 1, i.e. we put $i := i + 1$, and if $i \leq r$ we go back to step 3.
5. The multiplicities of D are the effective multiplicities of D'.

In the fourth chapter we define elementary transformations, quadratic transformations of the second and third kind. Then we see how these transfom plane curves, and finally we give Noether's theorem, that states that every reduced curve may be transformed, by a finite number of elementary quadratic transformations, in a curve with only ordinary singularities.

We see the simplest example of Cremona transformation of the plane that is not an automorphism.

Definition 0.4. *The rational application* $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *defined by:*

$$\alpha([x_0 : x_1 : x_2]) = [x_1x_2 : x_0x_2 : x_0x_1] \tag{13}$$

is called the quadratic elementary $transformation$. *The points:*

$$P_1 = [1 : 0 : 0], \quad P_2 = [0 : 1 : 0], \quad P_3 = [0 : 0 : 1]$$

are called the $fundamental$ *points of* α, *while the lines:*

$$l_1 = \overline{P_2P_3} = \{x_0 = 0\}\,, \quad l_2 = \overline{P_1P_3} = \{x_1 = 0\}\,, \quad l_3 = \overline{P_1P_2} = \{x_2 = 0\}$$

are called exceptional lines. From the definition it follows that α *is a morphism on* $\mathbb{P}^2 \setminus \{P_1, P_2, P_3\}$ *and an isomorphism on* $\mathbb{P}^2 \setminus \{l_1 \cup l_2 \cup l_3\}$, *on which* α *can be written:*

$$\alpha([x_0 : x_1 : x_2]) = \left[\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right].$$

Then α *is birational and* $\alpha^2 = \alpha \circ \alpha$ *is the identity automorphism of* $\mathbb{P}^2$.

The image under α of a straight line not passing through fundamental points is a conic for P_1, P_2 and P_3. More precisely, the net of lines of the plane (domain) is transformed into the net of conics (of codomain) passing through P_1, P_2 and P_3 and vice versa. This explains why the term "quadratic" to indicate this Cremona transformation.

Definition 0.5. *We say quadratic transformation a birational application of projective plane into itself which transforms the net of lines of the domain in a net of conics of codomain, whose general curve is irreducible. Suppose that the net of conics is defined by the passage through three points not aligned P_1, P_2 and P_3. Then the birational application α associated with Γ is called elementary quadratic transformation with fundamental points P_1, P_2 and P_3, and we write $\alpha = c(P_1, P_2, P_3)$ where c means "Cremonian". Finally we say that $l_1 = \overline{P_2P_3}$, $l_2 = \overline{P_1P_3}$, $l_3 = \overline{P_1P_2}$ are the exceptional lines of α*

We observe that each elementary quadratic transformation $c(P_1, P_2, P_3)$ is of the form 13, up to automorphism of the plane. In fact, if $\phi : \mathbb{P}^2 \to \mathbb{P}^2$ is a projectivity such that:

$$\phi(P_1) = [1:0:0], \quad \phi(P_2) = [0:1:0], \quad \phi(P_3) = [0:0:1],$$

then occurs immediately that:

$$c(P_1, P_2, P_3) = \phi^{-1} \circ \alpha \circ \phi$$

where α is the quadratic transformation defined by the formula 13.

Let $\alpha = c(P_1, P_2, P_3)$ be an elementary quadratic transformation. Now we see how to interpret α with blowing up. We consider the blowing up $\sigma : S \to \mathbb{P}^2$ of the points P_1, P_2 and P_3, and let $\sigma' : S' \to \mathbb{P}^2$ be a copy of σ. Then there exists an isomorphism such that $\alpha = c(P_1, P_2, P_3)$, i.e. the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\psi"] \arrow[d, "\sigma"'] & S' \arrow[d, "\sigma'"'] \\ \mathbb{P}^2 \arrow[r, "\alpha"] & \mathbb{P}^2 \end{tikzcd}$$

In particular, indicated with $\tilde{l}_i$ the strict transforms of the lines l_i in S, we have:

$$\psi(E_i) = \tilde{l}'_i, \quad \psi(\tilde{l}_i) = E'_i, \quad for\ i = 1, 2, 3,$$

where E'_i and $\tilde{l}'_i$ are the copies in S' respectively of E_i and $\tilde{l}_i$ (see Figure 0.7).

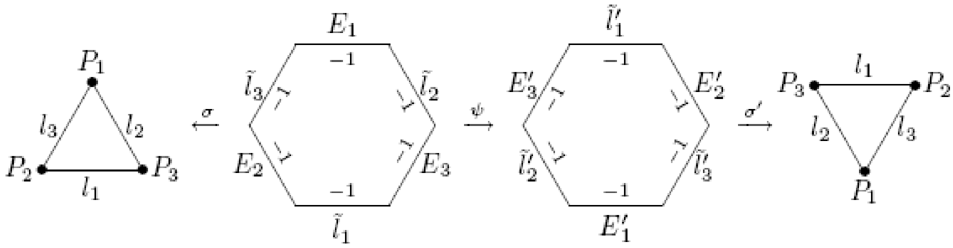

Figure 0.7: Elementary quadratic transformation

In other words, $\alpha = c(P_1, P_2, P_3)$ is the blowing up of P_1, P_2 and P_3 and the contraction $\tau = \sigma' \circ \psi$ of $\tilde{l}_1, \tilde{l}_2$ and $\tilde{l}_3$. We observe that $\tau(E_i) = l_i$ for $i = 1, 2, 3$, and this motivates the term "exceptional" for the lines l_i there. Then we see how this are used for plane curves. Let P be a singular point of a curve C of multiplicity $a > 1$. We recall that the singularitiy of C in P is called *ordinary* if P is proper and the strict transform of C in S, where $\sigma : S \to \mathbb{P}^2$ is the blowing up of P, intersects the exceptional curve E_P in a distinct points. Moreover, in this case, the intersections of the strict transform of C with E_P are transversal.

Then we see what is the strict (and total) transform of a plane curve according to an elementary quadratic transformation. Finally we show the "resolution of singularities " of a plane curve by Noether's method.

Theorem 0.4. *Every reduced curve may be transformed, by a finite number of elementary quadratic transformations, in a curve with only ordinary singularities.*

Then we analize the quadratic transformations of the second kind. We can consider three points P_1, P_2, P_3 such that:

$$P_1, P_2 \in \mathbb{P}^2, \quad P_3 >^1 P_1 \quad and \quad P_3 \notin \overline{P_1P_2}.$$

Then the conics passing through P_1, P_2 and P_3 form a network to which is associated a *quadratic transformation*, always indicated with $c(P_1, P_2, P_3)$ and said of *the second type.*

Lemma 0.3. *A quadratic transformation of the second type is the composition of two elementary quadratic transformations.*

Then we have last type of quadratic transformation. We suppose that P_1, P_2 and P_3 be three points on $\mathbb{P}^2$ such that:

$$P_3 >^1 P_2 >^1 P_1 \in \mathbb{P}^2, \quad P_3 \not\odot P_1 \text{ and } P_3 \not\subset \overline{P_1P_2}.$$

Then the conics passing through P_1, P_2 and P_3 form a net of irreducible conics defining a *quadratic transformation*, called of the *third kind*,

Lemma 0.4. *A quadratic transformation of the third type is the composition of four elementary quadratic transformations.*

We affirm that every quadratic transformation is one of the three types studied, up to projectivities.

In the fifth chapter we define the concept of simplicity of a curve that will be used to demonstrate Noether and Castelnuovo theorem, that states that every Cremonian transformation is the composition of an automorphism and a finite number of elementary quadratic transformation.
A Cremonian plane transformation is a birational application of the projective plane in itself. The set of Cremonian (plane) transformations is of course a group with the operation of composition and is called the *group of Cremona*. We start with the analysis of some properties of the Cremonian transformations.

Let $\gamma : \mathbb{P}^2 \to \mathbb{P}^2$ be a Cremonian transformation. The image $\gamma_*(R)$ of a generic line R of the plane is an irreducible rational curve. Moreover, the image $\Gamma = \gamma_*(|R|)$ of the net $|R|$ of the lines in the plane is an irreducible net of curves of geometric genus zero, that is a complete linear system of dimension 2 whose generic curve $\gamma_*(R)$ is irreducible and rational. We recall the notion of basis point of Γ and then we give some examples of Cremonian transformation. When there is a Cremonian transformation such that Γ has a basis point of multiplicity $d-1$, it is said $De\ Jonquieres\ transformation.$

Remark 0.3. *The transformations of De Jonquiere, can be considered as a natural generalization of the quadratic transformations*

What we now introduce is the concept underlying the proofs by induction of all the main theorems of this chapter that follow: the definition of semplicity of a curve.

Let D be an effective divisor S of degree d and multiplicity a_i at the blown up points P_i, for $i = 1, ..., r$. We reorder the points $P_1, ..., P_r$ according to multiplicity descending:

$$a_0 \geq a_1 \geq \cdots \geq a_{r-1} \geq a_r, \tag{14}$$

that is $a_j \leq a_i$ if $j > i$, and we add two fictitious points P_{-1} and P_{r+1} such that:

$$a_{-1} = \infty, \qquad a_{r+1} = -\infty.$$

We associate to the divisor D a set of parameters (k, h, s) defined by:

$$k = d - a_0, \qquad a_h > \frac{k}{2} \geq a_{h+1}, \tag{15}$$

and s is the number of satellites points between $P_0, ..., P_h$. We say that (k, h, s) are the *parameters of simplicity* of D. If D' is another effective divisor to which are associated the parameters (k', h', s'), then we say that D is simpler to D' if (k, h, s) is less than (k', h', s') in the lexicographic order:

$$(k, h, s) < (k'h's') \Longleftrightarrow \begin{cases} k < k' & or \\ k = k',\ h < h' & or \\ k = k',\ h = h',\ s < s' \end{cases}$$

We observe that k, h and s are integers, with $-1 \leq h \leq r$, for the definition of fictitious points. Also, if D is pure effective, then we have $k \geq 0$.

Through some lemmas we arrive to enunciate the important theorem of this last chapter.

Theorem 0.5. *A Cremonian transformation $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which is not a projectivity, is the composition of a finite number of elementary quadratic transformations. Expressly, if γ is not an automorphism, then:*

$$\gamma = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$$

where α_i is an elementary quadratic transformation , for $i = 1, ..., n \geq 1$.

Noether gave a first proof of this theorem stating that by applying a quadratic transformation we could lower the degree of the given Cremona transformation. But this demonstration remained incomplete. The Castelnuovo's method involved the use of a De Jonquieres transformation but it was not clear whether this transformation was well defined. These objections are overcome by using the concept of simplicity.

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