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The Clifford-Klein space form problem.

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Contents

Synthesis **2**

Bibliography **19**

Synthesis

The aim of this thesis is to present, and to describe solutions in same cases, an important problem posed by the mathematician Wilhelm Killing $(1847 - 1923)$. It consists in the ambitious study and classification of Riemannian complete and connected manifolds of constant curvature, called by Killing *space* $forms$, because they represented models for the idea of physical space at that time. Killing calls this the $Clifford - Klein$ *space form problem*, in honour of the efforts which the two mathematicians put on this problem.
As a matter of fact, he who has pushed a way to these studies and who has revolutionized the idea of space, he is B. Riemann in his fundamental paper "On the Hypotheses which lies at the Foundation of Geometry". William Kingdon Clifford $(1845 - 1879)$ and Felix Klein $(1849 - 1925)$ live and work in the latter half of the XIX century, which is rightly called the "golden age" of mathematics. Geometry as a science is born in ancient Greece and reaches its apex in Euclid' s masterpiece *The Elements*; for nearly 2000 years after that no many conceptional progresses are made on this subject. Even though it is possible to find progresses among the Arab, in the Europe of Renaissance and some reviviscences of pure geometry also in the period of French Revolution with Gaspard Monge $(1746 - 1818)$ and Lazare Carnot $(1753 - 1823)$, in Newton' s Principles, geometry of space is still the Euclidean one and only in XIX we see a nearly explosive recovery of geometry as a vital branch of mathematics.
In 1829, the essay "O načalach geometrii" (On the Principles of Geometry) written by Nicolaj Ivanovič Lobačevskij $(1793 - 1856)$ indicates the official

birth of non-euclidean geometry, that is maybe the most representative symbol of this new age of geometry, so that Lobačevskij is considered like the "Copernicus of Geometry" [3]. In spite of this, it needs to say that very similar concepts have been enunciated, nearly in the same time, by the great Carl Friedrich Gauss $(1777 - 1855)$ and by the hungarian mathematician János Bólyai $(1802 - 1860)$. [5]

For some decennia the non-euclidean geometry continues to represent a secondary aspect of mathematics, until when the general conceptions of G. F. Bernhard Riemann $(1826 - 1866)$ consecrate the non-euclidean geometry as an important part of mathematics. Riemann has a good education, he studies in Berlin and in Gottingen where he takes the degree.

In 1854 he becomes "Privatdozent" at Univesity of Gottingen with the statement of one of the most famous thesis in the history of mathematics: "Ueber die Hypothesen welche der Geometrie zu Grunde liegen" (On the Hypotheses which lies at the Foundation of Geometry). In this thesis we find a global vision of geometry as a study of $manifolds$ of any number of dimensions and in any kind of space . The Riemann' s geometries are non-euclidean in a more general way than in Lobačevskij' s sense. According to Riemann' s idea, the geometry would not talk necessarily about points, lines and space in the ordinary way, rather it has to study set of ordinate n-folds associated by means of any rules, following the idea of *measurement*.

Riemann attributes the difficulties encountered in the study of non-euclidean geometry to the fact that geometers had never separated what we now call the topological properties of space from its metric properties, and he proposed to distinguish them. Moreover he promises to show how different metric structures can be put on this new idea of space, so that one cannot possibly expect to deduce the parallel postulate of Euclid from topological considerations alone.

Now we want to expose the main results expressed in Riemann' s work until now examinated, thus the concept of $manifold$, the definition of *distance* and of the *curvature* of a manifold. A manifold of dimension n is a set of n

ordered real numbers $(x_1, x_2, \ldots, x_n)$.
About the operations of meausurement Riemann says:"*...Measuring involves the superposition of the quantity to be used as a standard for the others. Otherwise, one can compare two quantities only when one is part of the other, and then only as to "more" or "less", not as to "how much"...*" [15].
So he gives the definition of *distance* considering the lenght of lines, and looking to the quantities x and dx, the increment in the quantities x, he obtains the formula $ds = \sqrt{\sum (dx)^2}$. He finds the latter expression in the simpler case of the plane and of the space (euclidean), which he calls $flat\ manifolds$. But how do we can extend this to a $curved\ manifold$?
Riemann considers

$$(x_1, x_2, \ldots, x_n), \qquad (x_1 + dx_1, x_2 + dx_2, \ldots, x_n + dx_n)$$

as the coordinates of two infinitely near points of a generic manofold. So he generalizes the Pithagora' s Theorem and in this way, examining for example the case of dimension $n = 4$, he has that the distance between these two points is the square root of [1]

$$\begin{aligned} g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 \\ +g_{12}dx_1dx_2 + g_{13}dx_1dx_3 + g_{14}dx_1dx_4 \\ +g_{23}dx_2dx_3 + g_{24}dx_2dx_4 \\ +g_{34}dx_3dx_4, \end{aligned}$$

where $g_{11}, \ldots, g_{34}$ are functions of $x_1, \ldots, x_n$. Every time we do a specific choice of the letters g, we are defining a space.
From a generalization of the common experience, Riemann (and before him Gauss) takes his idea of *curvature*. It can be intended as the measurement of how much a curve departes from a line, which has zero curvature. In the same way for the surfaces, but in relation to a plane (of zero curvature too). Nevertheless Riemann succeeds in expressing the measure of the curvature in a point entirely in terms of all the g, in the general case of a space in n dimensions. This expression is called the *measure of the curvature of the*

space.

At the end, in other words, which results do we draw from the Riemann' s revolution in geometry? Surely the creation of a non-limited number of spaces and of geometries; moreover our concept of space is more clear.

Riemann' s work theaches to mathematicians to not believe in any geometry, in any absolute space. It is the first step towards the abolition of the "absolutes" of physics of the XIX century. Riemann' s results about metric and curvature have had a physic interpretation in the Theory of Relativity. Thus the real revolution of the scientific thought, caused by the Relativity would not has been possible without the fundamental Riemann' s contribution.

Shortly after Riemann' s memoir many geometries are found which could replace Euclidean vision of physical spaces. It is studying one of those examples, given by Clifford, that Killing folrmutes the so-called Clifford-Klein problem: to study Riemannian manifolds of constant curvature.

After this important historical picture it is time to analyse the development of the thesis. We start CHAPTER I with some remarks on the well-known concepts of differentiable manifolds and vector fields introducing the concept of *vector bundle*. A vector bundle is a five-tuple

$$\xi = (E, \pi, M, \oplus, \odot)\,,$$

where E and M are spaces (the "'total space"' and "'base space"' of ξ, respectively), with a continuos surjective map $\pi : E \to M$, and $\oplus$ and $\odot$ are maps which make each fibre $\pi^{-1}(x)$ into a n-dimensional vector space over $\mathbb{R}$, such that the following local triviality condition is satisfied: for each $x \in M$, there is a neighborhood U_x and a homeomorphism $t : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^n$ which is a vector space isomorphism from each $\pi^{-1}(y)$ onto $y \times \mathbb{R}^n$. When M is a manifold, the bundle $\pi : TM \to M$ is called the *tangent bundle*, where $TM = \bigcup_{x\in M} M_x$ is the disjoint union of the tangent spaces M_x at $x \in M$. A *section* of a bundle is a continuos function $s : M \to E$ such that $\pi \circ s =$ identity of M and we can conclude that a vector field on a manifold M is nothing less than a section of the tangent bundle TM.

Starting from a vector bundle we can obtain a new vector bundle over the same base space, simply replacing each fibre by some other vector space. In this way we obtain the *dual bundle* $\pi' : E' \rightarrow M$, where

$$E' = \bigcup_{x \in B} \left[\pi^{-1}(x)\right]^* .$$

When this construction is applied to the tangent bundle TM of M, the resulting bundle, denoted by T^*M is called the *cotangent bundle* of M. Sections of T^*M are the *covariant vector fields*, sections of TM are called *contravariant vector field*, they are also called covariant and contravariant tensors (or tensor fields) of order 1.

Thus if $T : V_1 \times \ldots \times V_m \rightarrow \mathbb{R}$ is a *multilinear function* and if $V_1, \ldots, V_k = V$, we denote the set of all such T by $\mathcal{T}^m(V)$. We can obtain the vector bundle $\pi : E \rightarrow B$, this time with

$$E' = \bigcup_{x \in B} \mathcal{T}^k\left(\pi^{-1}(x)\right).$$

The section of the bundle $\mathcal{T}^k(TM)$ are called *covariant tensor fields of order* k.

A *contravariant tensor field of order* k is just a section of the bundle $\mathcal{T}^k(T^*M)$.

At this point we look a a special kind of covariant tensor fields, an element $T \in \mathcal{T}^k(V)$ is called *alternating* if

$$T(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = 0 \qquad \text{if} \quad v_i = v_j \quad (i \neq j).$$

We denote by $\Lambda^k(V)$ the set of all alternating $T \in \mathcal{T}^k(V)$. Considering once again vector bundles we replace each fibre $\pi^{-1}(x)$ with $\Lambda^k(\pi^{-1}(x))$, and we obtain the *exterior k bundle*

$$\Lambda^k(TM) = \bigcup_{x \in M} \Lambda^k(M_x).$$

Its sections are just alternating covariant tensor fields of order k, are called *differential forms of degree* k on M.

A *Lie group* G is a differentiable manifold endowed with a group structure such that the map $G\times G\to G$ defined by $(g,h)\to gh^{-1}$ is differentiable. We define a differentiable *action of a Lie group* G on a differentiable manifold M as a differentiable map $F:G\times M\to M$ such that $g\to F(g,\cdot)$ is a homomorphism of G into the group of diffeomorphisms of M. A fundamental example of Lie group is $GL(n,\mathbb{R})$ the group of all $n\times n$ non-singular real matrices. If we consider on $GL(n,\mathbb{R})$ the product $[X,Y]=XY-YX$, this satisfies the axioms

$$[X,Y]=-[Y,X] \qquad \text{and}$$

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0.$$

An algebra whose product satisfies such axioms is called a *Lie algebra*.
For any $x\in M$ an ordered basis of the tangent space M_x, it is a $frame$ at x. Let an ordered n-tuple of vector fields on an open set $U\subset M$ (whose value at every point $x\in U$ forms a frame) $X=\{X_1,\ldots,X_n\}$, it is called a *moving* $frame$.
The moving frame on all open sets $U\subset M$ put a differentiable structure on B. The manifold B has a natural projection map

$$\pi:B\to M$$

such that $X:U\to B$ is a differentiable map such that $\pi\cdot X$ is the identity on U. So that exists a cover $\{U_\alpha\}$ of M such that $\pi^{-1}(U_\alpha)$ is diffeomorphic to $U_\alpha\times GL(n,\mathbb{R})$. The Lie group $GL(n,\mathbb{R})$ has a differentiable action on $B:B$ is called the $frame$ *bundle* on M.
We define a *connection* $H=\{H_b\}_{b\in B}$ on B as a choice of a subspace $H_b\subset B_b$, such that $GL(n,\mathbb{R})$ preverses the following decomposition $B_b=H_b+V_b$ of the total space in horizontal plus vertical space. Let X be a differentiable vector field on an open subset U of B, we have a unique decompositon $X=hX+vX$, where hX and vX are respectively horizontal and vertical fields. Let X_x be a vector at a point $x\in M$.
Given a point $b\in\pi^{-1}(x)$, consider the projection $\pi_*:B_b\to M_x$; it has kernel V_b, so $\pi_*:H_b\cong M_x$. Thus there is a unique horizontal vector $X_b^\lambda\in B_b$ which the map π_* sends to the vector X_x. Recall that each point $b\in\pi^{-1}(x)$

is a frame, then rightly we consider a horizontal vector field X^{λ} along the *fibre* $\pi^{-1}(x)$, and we call it the *horizontal lift* of X_x.
Now let $\sigma(t)$, $t_1 \leq t \leq t_2$, be a differentiable curve from $x \in M$ to $z \in M$. If $b \in \pi^{-1}(x)$ then σ has a unique horizontal lift $\widetilde{\sigma}_b$, such that $\widetilde{\sigma}_b(t_1) = b$, and $(b,t) \longrightarrow \widetilde{\sigma}_b(t)$ is differentiable on $\pi^{-1}(x) \times [t_1, t_2]$.
Define

$$\tau : \pi^{-1}(x) \longrightarrow \pi^{-1}(z) \quad \text{by} \quad \tau(b) = \widetilde{\sigma}_b(t_2).$$

Then τ is a diffeomorphism, $\tau(bg) = \tau(b)\, g$ for every $g \in GL(n, \mathbb{R})$, and there is a unique vector space isomorphism $\tau_0 : M_x \longrightarrow M_z$ such that

$$\tau\{b_1, \ldots, b_n\} = \{\tau_0(b_1), \ldots, \tau_0(b_n)\}.$$

τ and τ_0 defined above, are the operation of parallel translation, respectively of frames and vectors along σ, determined by the connection H on B.

So let $\sigma(t)$ be a differential curve in M, and suppose that we have a vector field $t \longrightarrow Y_t \in M_{\sigma(t)}$ along σ. For any two values t_1, t_2 of t, we have the parallel translation $\tau_{t_1,t_2} : M_{\sigma(t_1)} \longrightarrow M_{\sigma(t_2)}$ along $\sigma|_{[t_1,t_2]}$.
We can say that the vector field $Y = \{Y_t\}$ is *parallel along* σ if $\tau_{t_1,t_2}(Y_{t_1}) = Y_{t_2}$ for all t_1, t_2. Now we define functions s^i and y^i by: $\sigma'(t) = \sum s^i(t)\, X_{i\sigma(t)}$ and $Y_t = \sum y^i(t)\, X_{i\sigma(t)}$; and we consider τ_h, the parallel translation of vectors along σ from $\sigma(t)$ to $\sigma(t+h)$. The difference quotient $\nabla_{\sigma'(t)} Y \equiv \lim_{h \to 0} \frac{1}{h}\{\tau_{-h} Y_{t+h} - Y_t\}$ is given by

$$\nabla_{\sigma'(t)} Y = \sum_i \left\{ \left(dy^i + \sum_k y^k \bar{\omega}^i_k \right) (\sigma'(t)) \right\} X_{i\sigma(t)},$$

where $\bar{\theta}^i$ are linear differential forms on U defined by $\bar{\theta}^i(X_j) = \delta^i_j$, and $\bar{\omega}^i_k = \sum \Gamma^i_{jk} \bar{\theta}$, where the functions Γ^i_{jk} on U are the *Christoffel symbols* (the components of the connection relative to the moving frame X).
More generally, a vector $\nabla_Z Y \in M_x$ is defined whenever $Z \in M_x$ and Y is

a smooth vector field on a neighborhood of x. $\nabla_Z Y$ is called the *covariant derivative* of Y along Z.
Once a connection is given on the bundle B, if Y and Z are vector fields on an open set $U \subset M$, then we define the fundamental tensors:

$$\begin{aligned} T(Y,Z) &= \nabla_Y Z - \nabla_Z Y - [Y,Z], \quad \text{the torsion tensor;} \\ R(Y,Z) &= \nabla_Y \nabla_Z - \nabla_Z \nabla_Y - \nabla_{[Y,Z]}, \quad \text{the curvature tensor.} \end{aligned}$$

In CHAPTER II we introduce Riemannian manifolds: they are differentiable manifolds endowed with a metric. If M a differentiable manifold, a $pseudo - riemannian\ metric$ on M is a differentiable field $g = \{g_x\}_{x \in M}$ of nondegenerate symmetric bilinear forms g_x on the tangent spaces M_x of M. In the language of bundles: let $X = \{X_1, \ldots, X_n\}$ be a moving frame on an open set $U \subset M$, and let $\{\theta^i\}$ be the dual coframe defined by $\theta^i(X_j) = \delta_i^j$. Then we define differentiable functions on U by $g_{ij}(x) = g_x(X_{ix}, X_{jx})$ and see the traditional expression

$$g = \sum_{i,j} g_{ij} \theta^i \otimes \theta^j.$$

The Fundamental Theorem of Riemannian Geometry says:

Theorem 0.1. *The frame bundle of a pseudo-riemannian manifold M has a unique connection for which: (i) parallel translation of tangent vectors along any curve in M preserves inner products, and (ii) the torsion tensor $T = 0$. This connection is called the $Levi - Civita$ connection.*

From now on all affine concepts already seen like for example parallelism and curvature, refer to this connection. One of the main instruments of differential geometry is surely the Gaussian curvature. In "Disquisitiones generales circa superficies curvas" (1828), considering a surface $S \subset \mathbb{R}^3$, Gauss looks, at any point $x \in S$, the "difference" between S and the tangent space in x S_x as a real number, the *Gaussian curvature*. In particular let $<,>$ be the standard inner product defined on each tangent plane S_x, and let $x : S \to \mathbb{R}^3$ be the inclusion, viewed as position vector on S. Let ξ denote a smooth choice of unit normal on S, defined up to sign by $< \xi_x, S_x >= 0$ and $< \xi_x, \xi_x >= \pm 1$. Then the first and second fundamental forms on S are given by

$$I = ds^2 =< dx, dx > \quad \text{and} \quad II = - < d\xi, dx > .$$

We write $||v||^2$ for $< v, v >$ and say that v is "nondegenerate" if $||v^2|| \neq 0$. If $v \in S_x$, we have $I(v, v) = ||v||^2$, and if, further, v is nondegenerate, we can define the remember the definition of the "normal curvature" $k_n(v)$ as $II(v, v) = ||v||^2 k_n(v)$.

If P is the plane through x spanned by ξ_x and v, then $k_n(v)$ is the signed curvature at x of the plane $S \cap P$ in P. We know that if k_n takes extremal values on S_x (which is automatic if I is positive or negative definite),we can call them "principal curvatures". There are just two of them (for different directions), and their product $K = k_1 k_2$ is the determinant of II divided by the signed determinant of I, it is the Gaussian curvature.

The inner product in $\mathbb{R}^3$ $<,>$ defines by restriction an inner product on the tangent spaces of S, this gives S a natural structure of Riemannian manifold. Now, considering a generic pseudo-riemannian metric, let $\{v_1, v_2\}$ be a moving frame on an open subset of S and let $\{\alpha^1, \alpha^2\}$ the dual form, we can express these ideas above in other words. So relative to the frame,

$$I = \sum g_{ij} \alpha^i \otimes \alpha^j \quad \text{and} \quad II = \sum b_{ij} \alpha^i \otimes \alpha^j$$

and the Gaussian curvature is

$$K_c = det(b_{ij}) / det(g_{ij}) ||\xi||^2.$$

Riemann is able to generalize this concept passing form two to n dimension with the definition of sectional curvature.

Let $x \in M$, a pseudo-riemannian manifold with metric g. Suppose that E is a nondegenerate plane section (2-dimensional linear subspace) of the tangent space M_x. We take $U \subset E$, if it is a sufficiently small neighborhood of 0 in E, then (i) $\exp_x : U \to \exp_x(U) = S_E$ is a diffeomorphism, so S_E is a "local" surface containing x and with $(S_E)_x = E$, and (ii) if $z \in S_E$ then

$(S_E)_z$ is a nondegenerate subspace of M_z. Then the inner products on M_z induce a pseudo-riemannian metric on the surface S_E. We define the *sectional curvature* $K(x, E)$ to be the Gauss curvature of S_E at x,

$$K(x, E) = K_{S_E}(x).$$

Theorem 2.5 shows that the sectional curvature is a kind of normalization on the plane section of the curvature tensor on the manifold, thus

$$K(x, E) = -\frac{g(R(Y, Z)Y, Z)}{g(Y, Y)\, g(Z, Z) - g(Y, Z)^2}.$$

Given a number K and a point $x \in M$, we say that M has *constant curvature K at x* if $K(x, E) = K$ for every nondegenerate plane section $E \subset M_x$. Now it can happen that the value of constant curvature is the same K for each point of the manifold, so *M has constant curvature K* if M has constant curvature at each of its points.

In CHAPTER III we denote with $\mathbf{R}^n_s$, $0 \leq s \leq n$, denote the vector space of real n-tuples $x = (x^1, \ldots, x^n)$ with the bilinear form

$$\mathbf{b}^n_s(x, y) = -\sum_{i=i}^{s} x^i y^i + \sum_{j=s+1}^{n} x^j y^j.$$

Let $\mathbf{g}^n_s$ be the pseudo-riemannian metric on $\mathbf{R}^n_s$ obtained by euclidean-parallel translation of the form $\mathbf{b}^n_s(x, y)$. $\mathbf{R}^n_s$ is a complete simply connected pseudo-riemannian manifold of signature $(s, n-s)$ (from the $\mathbf{b}^n_s(x, y)$) and of constant curvature zero. After we find nonzero curvature models:

$$\mathbf{S}^n_s = \left\{x \in \mathbf{R}^{n+1}_s : \mathbf{b}^{n+1}_s(x, x) = r^2\right\}$$

$$\mathbf{H}^n_s = \left\{x \in \mathbf{R}^{n+1}_{s+1} : \mathbf{b}^{n+1}_{s+1}(x, x) = -r^2\right\}.$$

are complete pseudo-riemannian manifolds of signature $(s, n-s)$ and respective constant curvature r^{-2} and $-r^{-2}$, the *pseudo$-$riemannian spheres* and the *hyperbolic spaces*.

We define :

$\tilde{\mathbf{S}}^n_s = \mathbf{S}^n_s$ for $s \neq n-1, n$;
$\tilde{\mathbf{S}}^n_{n-1}$: universal pseudo-riemannian covering manifold of $\mathbf{S}^n_{n-1}$;
$\tilde{\mathbf{S}}^n_n$: component of $(0, \ldots, 0, 1)$ in $\mathbf{S}^n_n$.
Similarly we define $\tilde{\mathbf{H}}^n_s = \mathbf{H}^n_s$ for $s \neq 0, 1$;
$\tilde{\mathbf{H}}^n_1$: universal pseudo-riemannian covering manifold of $\mathbf{H}^n_1$;
$\tilde{\mathbf{H}}^n_0$: component of $(1, 0, \ldots, 0)$ in $\mathbf{H}^n_0$.
Now $\tilde{\mathbf{S}}^n_s$ and $\tilde{\mathbf{H}}^n_s$ are complete simply connected manifolds, thus we can apply the important Corollary 2.10. It says that a pseudo-riemannian manifold M is complete and of constant curvature K, if and only if M is isometric to the quotient N/Γ of N by a free properly discontinuos group of isometries; Where N is a complete simply connected pseudo-riemannian manifold of constant curvature K, and M has a tangent space isometric to a tangent space of N (i.e., same dimension and signature of metric). We say that Γ acts freely on N when the set $\{\gamma \in \Gamma : \gamma(x) = x\}$ is trivial. Γ is called *properly discontinuos on* N if every point $x \in N$ has a neighborhood U such that $\{\gamma \in \Gamma : \gamma(U) \cap U\}$ is finite.
Finally

Theorem 0.2. *Let M^n_s be a pseudo-riemannian manifold of signature $(s, n-s)$, $0 \leq s \leq n$ and $n \geq 2$, then M^n_s is complete, connected and of constant curvature K, (a real number) if and only if it is isometric to a quotient*

$$\tilde{\boldsymbol{S}}^n_s/\Gamma \quad \text{if} \quad K > 0, \quad \boldsymbol{R}^n_s/\Gamma \quad \text{if} \quad K = 0, \quad \tilde{\boldsymbol{H}}^n_s/\Gamma \quad \text{if} \quad K < 0$$

where Γ is a group of isometries acting freely and properly discontinuosly.

In our main interest of the riemannian case there is the following Killing-Hopf corollary

Corollary 0.1. *Let M^n be a riemannian manifold of dimension $n \geq 2$ and let K be a real number. Then M^n is complete, connected and of constant curvature K, if and only if it is isometric to a quotient*

$$
\begin{array}{llrll}
\boldsymbol{S}^n/\Gamma & \textit{with} & \Gamma \subset \boldsymbol{O}\left(n+1\right), & \textit{if} & K>0 \\
\boldsymbol{R}^n/\Gamma & \textit{with} & \Gamma \subset \boldsymbol{E}\left(n\right), & \textit{if} & K=0 \\
\boldsymbol{H}^n/\Gamma & \textit{with} & \Gamma \subset \boldsymbol{O}^1\left(n+1\right), & \textit{if} & K<0
\end{array}
$$

Where $\mathbf{E}\left(n\right)$, $\mathbf{O}\left(n+1\right)$, $\mathbf{O}^1\left(n+1\right)$ are the groups of all isometries respectively of $\mathbf{R}^n$, $\mathbf{S}^n$ and $\mathbf{H}^n$.
These models are the so called *spherical* (if $K>0$), *euclidean* (if $K=0$) or *hyperbolic* (if $K<0$) *space forms*. The problem of studying these "space forms" was first well-formulated by W. Killing, who gave the name "Clifford-Klein space form problem" to the problem of classifyng them.
In CHAPTER IV we describe the solution of this important problem in the fascinating case of *homogeneous* riemannian manifold.
Recall that more generally, a pseudo-riemannian manifold is called homogeneous if the full group of isometries is transitive: given $x, y \in M$, there exists $\gamma \in \Gamma$ (the full group of isometries) such that $\gamma\left(x\right)=y$.
Since the born of scientific thought until the Theory of Relativity there are two properties which have characterized the space, *isotropy* and *homogeneity*.
So consider Euclid' s postulates:
III. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.
IV. All right angles are congruent.
In postulate III the possibility of drawing a circle in any point, indicates a space with no preferred directions, then an isotropic space. Postulate IV says that there is a transitive action of a group of isometries which sends right angles in right angles, therefore it is talking about a homogeneous space.
Similarly looking at the "Cosmological Principle": *Viewed on sufficiently large distance scales, there are no preferred directions or preferred places in the Universe*;
we translate the isotropy of the space saying that it has constant curvature K. Like above, all points are undistinguishable (no preferred places): an uni-

form space which we call homogeneous.
Going in the middle of the question, we take an isometry f of a metric space S, and we consider the *displacement function* given by

$$\delta_f(x) \text{ is the distance from } x \text{ to } f(x).$$

We say that f is a *Clifford translation* if δ_f is constant. We show that an homogeneous manifold is complete, so from Killing' s models find in Corollary **??** and by Clifford translations we obtain our classification in the following theorem:

Theorem 0.3. *Let M^n be a connected homogeneous riemannian manifold of dimension n and constant curvature $K = 0$, then M^n is isometric to the product $\boldsymbol{R}^m \times \boldsymbol{T}^{n-m}$ of a euclidean space with a flat riemannian torus. If $K < 0$, then M^n is isometric to the hyperbolic space $\boldsymbol{H}^n$. If $K > 0$, then M^n is isometric to a manifold $\boldsymbol{S}^n/\Gamma$ where (i) $\boldsymbol{F}$ is a field $\boldsymbol{R}$ (real numbers), $\boldsymbol{C}$ (complex numbers) or $\boldsymbol{Q}$ (quaternions), (ii) $\boldsymbol{S}^n$ is the sphere $\|x\| = K^{-\frac{1}{2}}$ in a left hermitian vector space V over $\boldsymbol{F}$ where V has a real dimension $n+1$, (iii) Γ is a finite multiplicative group of elements of norm 1 in $\boldsymbol{F}$ which is not contained in a proper subfield $\mathbf{F_1}, \mathbf{R} \subset \mathbf{F_1} \subsetneq \boldsymbol{F}$, of $\boldsymbol{F}$, of $\boldsymbol{F}$, and (iv) Γ acts on $\boldsymbol{S}^n$ by $\boldsymbol{F}$-scalar multiplication of vectors. Conversely, all the manifolds listed are n-dimensional riemannian homogeneous manifolds of constant curvature K.*

We complete the case $K > 0$ obtaining:

Corollary 0.2. *A connected riemannian homogeneous manifold M^n of dimension n and constant curvature $K > 0$ is determined up to isometry by the fundamental group $\pi_1(M^n)$. The only cases are (i) $M^n = \boldsymbol{S}^n$; (ii) $M^n = \boldsymbol{P}^n$ (which is the real projective $n-space$ $\boldsymbol{S}^n/\{\pm I\}$; (iii) $n+1 \equiv 0$ modulo 2, while $M^n = \boldsymbol{S}^n/\mathbf{Z_m}$ with $m > 2$; and (iv) $n+1 \equiv 0$ modulo 4, while $M^n = \boldsymbol{S}^n/\mathbf{D}^*{}_m$ with $m > 2$ or $M^n = \boldsymbol{S}^n/\mathbf{T}^*$ or $M^n = \boldsymbol{S}^n/\mathbf{O}^*$ or $M^n = \boldsymbol{S}^n/\mathbf{I}^*$.*

Where $\mathbf{Z}_m$ is the cyclic group of the integers *modulo* m, $\mathbf{D}^*{}_m$ is the binary dihedral group and $\mathbf{T}^*$, $\mathbf{O}^*$, $\mathbf{I}^*$ are the binary polyhedral groups.

Recall that given an arcwise connected space S, an arc σ in S is *based at* $x \in S$ if $\sigma(0) = x = \sigma(1)$. Arcs σ, τ based at s are *homotopic* (rel. x) if there is a *homotopy*, a continuos map $G : [0,1] \times [0,1] \to S$, such that

$$\begin{aligned} G(t,0) = \sigma(t), \quad & G(t,1) = \tau(t) \quad \forall t, \\ \text{and} \quad G(0,s) = \sigma(0), \quad & G(1,s) = \tau(1) \;\; \forall s. \end{aligned}$$

This is an equivalence relation; the equivalence classes are the elements of the $fundamental\ group$ $\pi_1(S,x)$.

In CHAPTER V we consider flat riemannian manifolds. So we start from the quotient $\mathbf{R}^n/\Gamma$ with $\Gamma \subset \mathbf{E}(n)$ acting freely and properly discontinuosly on $\mathbf{R}^n$, and we find that they are flat tori $\mathbf{T}^n/\Gamma$ where Γ is a discrete uniform (compact quotient) subgroup of $\mathbf{E}(n)$ consisting of pure translations. After, there are the flat cylinders $\mathbf{R}^n/\Delta$ where Δ is a discrete group of pure translation of $\mathbf{R}^n$. Now if $\mathbf{R}^k \subset \mathbf{R}^n$ is the span of Δ, then $\mathbf{R}^k/\Delta$ is a flat torus and $\mathbf{R}^n/\Delta$ is isometric to $\left(\mathbf{R}^k/\Delta\right) \times \mathbf{R}^{n-k}$. Let M be a manifold with a connection on its frame bundle. We say that a manifold $N \subset M$ is *totally geodesic* if every geodesic of M tangent to N at a point is completely contained in N. Thus we see that every flat cylinder may be retracted onto a compact totally geodesic submanifold which carries much of the geometric informations concerning the cylinder. Theorem 5.6 generalize this retraction for all flat complete connected riemannian manifolds. So we devote our attention completely to compact flat manifolds. Given $x \in M$ and a sectionally smooth curve σ based at x, we have the parallel translation $\tau_\sigma : M_x \to M_x$ of tangent vectors along σ. These τ_σ form a group Ψ_x of linear transformations of M_x. Ψ_x is the *linear holonomy* group at x. For flat manifold the map $h : \pi_1(M,x) \to \Psi_x$ is called the *holonomy homomorphism*.

Let $p : S \to T$ a covering, and consider the homomorphisms $h : S \to S$, such that $p \cdot h = p$, they are called $deck\ transformations$. We define a *normal riemannian covering* as a covering $p : S \to T$ (with p a local isometry) of riemannian manifolds, such that, considering the map $p_* : \pi_1(S,s) \to \pi_1(T,t)$, we have that $p_*\pi_1(S,s)$ is a normal subgroup of $\pi_1(T,t)$ for every $s \in S$,

where $t = p(s)$.
Applying these ideas we have:

Corollary 0.3. *Let M^n be a connected compact flat n-dimensional riemannian manifold. Then there is a normal riemannian covering $p : \boldsymbol{T}^n \to M^n$, by a flat torus, in which the (necessarily finite) group of deck transformations is isomorphic to the linear holonomy group of M^n; so $M^n = \boldsymbol{T}^n / \Psi$.*

At this point it is natural to ask which finite groups occur as linear holonomy groups of flat riemannian manifolds. The surprising answer is given by the following Auslander-Kuranishi theorem:

Theorem 0.4. *Let G be any finite group. Then G is the linear holonomy group of some flat compact connected riemannian manifold.*

With this theorem we conclude our thesis.

Figura 0.1

Picasso, *Les Damoisselle d' Avignon*, 1907, New York, Museum of Modern Art.

"It is the first picture of the cubist art. Picasso, like Riemann, smashes the classical space, like Einstein, he starts a revolution, but in figurative arts."

FIgura 3.1

Picasso, *Violon et verre sur un table*, 1903, St. Petersburg, The State Hermitage Museum.

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