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Symplectic Geometry

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Two centuries ago symplectic geometry provided a language for classical mechanics. Symplectic structures first arose in the study of classical mechanical systems such as the planetary systems, and almost all the classical work on symplectic geometry was focused on the attempt to understand how these systems behave. Through its recent huge development, symplectic geometry conquered an independent and rich territory, as a central branch of differential geometry and topology.

To mention just a few key landmarks, one may say that symplectic geometry began to take its modern shape with the formulation of the Arnold conjectures in the 60's and with the foundational work of Weinstein in the 70's. A paper of Gromov in the 80's gave the subject a whole new set of tools: pseudo-holomorphic curves. Gromov also first showed that important results from complex Kähler geometry remain true in the more general symplectic category, and this direction was continued in the 90's in the work of Donaldson on the topology of symplectic manifolds and their symplectic submanifolds, and in the work of Taubes in the context of the Seiberg-Witten invariants.

This work is essentially divided in two parts, in the first two Chapters we study some notions of linear theory, while in the other three Chapters we introduce the nonlinear theory. Here is a description of the work chapter by chapter.

We begin, in Chapter 1, with the concept of symplectic vector space, which is a pair (V, ω) where V is a real vector space equipped with a skew-symmetric and nondegenerate bilinear form ω called linear symplectic structure, or symplectic form. We show, using a standard theorem for skew symmetric bilinear forms (Theorem 1.1.2), that a symplectic vector space must be necessarily of even dimension and we prove the existence of symplectic bases (Remark 1.1.3).

Therefore we introduce the notion of linear symplectomorphism. A linear symplectomorphism between two symplectic vector spaces is a linear isomorphism which preserves the symplectic structure. An important result is the following:

Proposition 1 (Remark 1.2.6)**.** *Every $2n$-dimensional symplectic vector space (V, ω) is symplectomorphic to the prototype $(\mathbb{R}^{2n}, \omega_0)$. A choice of a symplectic basis for (V, ω) yields a symplectomorphism to $(\mathbb{R}^{2n}, \omega_0)$.*

We conclude the Chapter by focusing attention on the subspaces of a symplectic vector space and their properties, with particular regard to Lagrangian subspaces. Lagrangian subspaces are closely related to symplectomorphisms, as we can see from:

Lemma 2 (Lemma 1.3.7)**.** *Let (V, ω) be a symplectic vector space and let $\Psi : V \to V$ be a linear map. Then Ψ is a linear symplectomorphism if and only if the graph*

$$\Gamma_\Psi := \{(v, \Psi(v)) \mid v \in V\}$$

is a Lagrangian subspace of the symplectic vector space $(V \oplus V, (-\omega) \oplus \omega)$; where $((-\omega) \oplus \omega)((u, v), (r, s)) = -\omega(u, r) + \omega(v, s)$ with $(u, v), (r, s) \in V \times V$.

In Chapter 2 we study linear symplectomorphisms of a symplectic vector space in more detail.

Since all symplectic vector spaces of the same dimension are symplectomorphic, it suffices to consider the case $V = \mathbb{R}^{2n}$ with the standard symplectic form ω_0. Thus we can identify a linear symplectomorphism with the matrix $\Psi \in GL_{2n}(\mathbb{R})$ which represents it. Such a matrix is said to be a symplectic matrix. We prove that symplectic matrices form a group, namely the symplectic linear group $Sp(2n)$ (Lemma 2.1.2). In particular, $Sp(2n) \leq SL_{2n}(\mathbb{R})$ with equality if and only if $n = 1$ (Remark 2.1.3).

Furthermore we study the relations between $Sp(2n)$ and the Siegel upper half space, which is the space of complex symmetric matrices with positive definite imaginary part.

Proposition 3 (Proposition 2.4.1)**.** *The homogeneous space $Sp(2n)/U(n)$ is diffeomorphic to the Siegel upper half space S_n and hence it is contractible.*

In Section 6 we prove the affine nonsqueezing theorem:

Theorem 4 (Theorem 2.6.2)**.** *Let $\varphi \in ASp(2n)$ such that $\varphi(B^{2n}(r)) \subset Z^{2n}(R)$. Then $r \leq R$.*

which states that a ball $B^{2n}(r)$ of radius r in $\mathbb{R}^{2n}$ can be embedded into a symplectic cylinder $Z^{2n}(R)$ of radius R by an affine symplectomorphism only if $r \leq R$. Thanks to this result we can characterize the symplectic and the anti-symplectic matrices. In particular we have:

Theorem 5 (Theorem 2.6.3)**.** *Let $\Psi \in GL_{2n}(R)$ such that Ψ and Ψ^{-1} have the nonsqueezing property. Then Ψ is either symplectic or anti-symplectic.*

Then we define the linear symplectic width of a subset of $\mathbb{R}^{2n}$. An important statement is the following.

Theorem 6 (Theorem 2.6.7)**.** *Let $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. Then the following are equivalent.*
(i) *Ψ preserves the linear symplectic width of the ellipsoids centred at 0.*
(ii) *The matrix Ψ is either symplectic or anti-symplectic.*

Finally we introduce complex structures on a real vector space. In particular we examine the complex structures on a symplectic vector space (V, ω) which are compatible with the symplectic form ω.

The third Chapter is dedicated to symplectic manifolds and their submanifolds.

A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form, that is a closed and nondegenerate differential 2-form on M. By Remark 3.2.2 symplectic manifolds are necessarily of even dimension and orientable (Remark 3.2.4).

The first example is $(M = \mathbb{R}^{2n}, \omega_0)$, with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$, and $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. A fundamental class of symplectic manifolds is given by the cotangent bundle of an n-dimensional manifold (Sections 3 and 4).

Therefore we study the submanifolds of a symplectic manifold, giving some examples of Lagrangian submanifolds. For instance we show that, if X is an n-manifold with cotangent bundle T^*X, then the zero section $X_0 \subset T^*X$ is Lagrangian (Example 3.6.2). For what concern the image of another section of the cotangent bundle, we have:

Proposition 7 (Proposition 3.6.3)**.** *The graph of a* 1*-form* μ *is a Lagrangian submanifold of the cotangent bundle* T^*X *of an* n*-manifold* X *if and only if* μ *is a closed form.*

The next theorem is the analogous for manifolds of Lemma 1.3.7.

Theorem 8 (Theorem 3.6.5)**.** *Let* (M_1, ω_1) *and* (M_2, ω_2) *be symplectic* $2n$*-manifolds,* $\pi_1 : M_1 \times M_2 \to M_1$ *and* $\pi_2 : M_1 \times M_2 \to M_2$ *be the projections, and let* $\tilde{\omega} = (\pi_1)^*\omega_1 - (\pi_2)^*\omega_2$*. Then*

1. $(M_1 \times M_2, \tilde{\omega})$ *is a symplectic manifold.*
2. *A diffeomorphism* $\varphi : M_1 \to M_2$ *is a symplectomorphism if and only if its graph* Γ_φ *is a Lagrangian submanifold of* $(M_1 \times M_2, \tilde{\omega})$*.*

The aim of Chapter 4 is to prove a theorem of Darboux:

Theorem 9 (Theorem 4.3.2)**.** *Every symplectic form* ω *on a manifold* M *is locally diffeomorphic to the standard symplectic form* ω_0 *on* $\mathbb{R}^{2n}$*. That is: for any* $p \in M$ *we can find a coordinate system* $(\mathcal{U}, x_1, \dots, x_n, y_1, \dots, y_n)$ *centered at* p *such that on* $\mathcal{U}$

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

The main ingredients in proving this important result are Moser's argument on the isotopy of symplectic forms (see the discussion at the beginning of section 3), and Moser theorem:

Theorem 10 (Theorem 4.3.1). *Let M be a manifold, X a submanifold of M and $i : X \hookrightarrow M$ the inclusion map. Let ω_0 and ω_1 be symplectic forms in M such that $\omega_0|_p = \omega_1|_p$ for all $p \in X$. Then there exist neighborhoods U_0, U_1 of X in M and a diffeomorphism $\varphi : U_0 \to U_1$ such that*

$$\varphi|_X = Id, \qquad \varphi^*\omega_1 = \omega_0.$$

The great importance of Darboux theorem is that it shows that symplectic geometry has no local invariant; this is a great contrast with Riemannian geometry where the curvature is a local invariant.

Finally, in Chapter 5, we extend the concept of complex structure to manifolds by defining almost complex structures. We also introduce the notion of compatible almost complex structures and we prove that:

Proposition 11 (Proposition 5.1.4). *Let (M, ω) be a symplectic manifold. Then there always exist almost complex structures J on M which are compatible with ω.*

This fact establishes a link from symplectic geometry to complex geometry and it is a point of departure for the modern techniques in symplectic geometry.

Hence we recall the notion of complex manifolds and we show that:

Proposition 12 (Proposition 5.2.2). *Any complex manifold has a canonical almost complex structure.*

In this context we give a brief description of Kähler manifolds which are important since the provide many example of symplectic manifolds. For some time, people wondered whether every symplectic manifold was in fact Kähler. Now we know that the situation is the following:

$$\begin{array}{ccc} \text{symplectic} & \Leftarrow & \text{Kähler} \\ \Downarrow & & \Downarrow \\ \text{almost complex} & \Leftarrow & \text{complex.} \end{array}$$

In general the opposite implications are false.

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