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Luca Dell'Anna

A Torelli Theorem in Graph Theory and Tropical Geometry

Relatore

Prof.ssa Lucia Caporaso

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The essence of mathematics lies in its freedom.
Georg Ferdinand Ludwig Philipp Cantor

Geometry is not true, it is advantageous.
Henri Poincaré

A child's first geometrical discoveries are topological. If you ask him to copy a square or a triangle, he draws a closed circle.
Jean Piaget

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Introduction

Tropical geometry is a very young branch of mathematics that is currently creating a brand new geometry over the so called tropical semifield. Many mathematicians are venturing on rewriting classical results of algebraic geometry using the new tropical language. Since the main objects studied by geometry are varieties, one of the first purposes of tropical geometry is to understand how tropical varieties behave and interact with each other, starting from the 1-dimensional case: tropical curves. Among all classic results of algebraic geometry of which we would like to have a tropical version, we are interested in the Torelli's Theorem. The statement of the theorem is the following:

Theorem. *Two curves C and C' are isomorphic if and only if the Jacobians of the curves C and C' are isomorphic as principally polarized abelian varieties.*

An alternative statement is:

Theorem. *The map*

$$\begin{aligned} M_g &\longrightarrow A_g \\ C &\longmapsto (Jac(C), \Theta(C)), \end{aligned}$$

from the moduli space of curves of genus $g \geq 1$, to a moduli space of principally polarized abelian varieties of dimension g, is injective.

Since is proved that tropical curves are, actually, metric graphs that are embedded into a tropical space, then our problem is to determine if Torelli's Theorem is true for this type of graphs. So, first of all, we need a graph theoretical version of the Torelli's Theorem. We'll prove that the theorem doesn't hold, in its stronger form, for general graphs, but only for a special class of graphs, the 3-edge connected graphs.
This thesis is entirely based on a 2009 work, [CV], by Lucia Caporaso and Filippo Viviani, published in 2010. The purpose of the thesis is to present the

article's main results, providing the reader with the basic theoretical tools of topology and graph theory needed for a full understanding of the paper.

In this summary we will present only the new and most important tools that were used to prove the Torelli Theorem assuming that the reader is confident with homology and graph theory.

Chapter 1

Homology and Graph Theory

In the first chapter of the thesis we introduced the reader to singular homology providing the basic theoretical instruments needed in order to determine the homology group of a graph. After that we focused our attention on graphs from the point of view of graph theory, introducing the concepts of connectivity, orientation, connectivization and C1-sets. These instruments have been used to prove the graph-theoretical version of the Torelli Theorem in Chapter 2.
For us, a *graph* will be always a finite *multigraph* that we define as:

Definition 1.0.1. A *multigraph* is a pair $G = (V, E)$ where:

- V is a set with elements called *vertices*;
- E is a *multiset* of unordered pairs of vertices, called *edges*: E consists of pairs of the form (u, v) with $u, v \in V$, where multiple copies of the same pair and elements of the form (v, v) can occour.

We say that two graphs, G and G' are *isomorphic* if there exists a bijection $\varphi : V \longrightarrow V'$ such that $uv \in E \iff \varphi(u)\varphi(v) \in E'$.
If G in connected and a vertex v is such that $G - v$ is not connected then v is a *separating vertex* or a *cutvertex*. Similarly, an edge e such that $G - e$ is disconnected is called a *separating edge* or a *bridge*. If a pair of non-separating vertices or edges disconnects a graph when removed, it will be called a *separating pair of vertices* or a *separating pair of edges* ; the set of all separating edges of a

graph is denoted $E(G)_{\text{sep}}$. If G is not connected we say that an edge or a vertex is separating if it's separating for a component of G.

Proposition 1.0.2. *e is a separating edge if and only if it doesn't lie on any cycle.*

A graph G is called *k-connected*, with $k \geq 1$, if G has at least $k+1$ vertices and $G-X$ is connected for every $X \subseteq V$ with $|X| < k$. So a graph is k-connected if we can remove any $k-1$ vertices keeping the graph connected. The greatest integer k such that G is k-connected is the *connectivity*, $\kappa(G)$, of G.
If G has at least 2 vertices and the graph obtained from G by deleting any $k-1$ edges is connected then G is said to be *k-edge connected*. The greatest integer k such that G is k-edge connected is its *edge connectivity*, $\lambda(G)$. Again $\lambda(G) = 0$ if G is disconnected, $\lambda(G) = 1$ if and only if G is connected and $\lambda(G) = 2$ if and only if G is connected and free from separating edges.

Proposition 1.0.3. *If G is non-trivial then $\kappa(G) \leq \lambda(G) \leq \delta(G)$. Where $\delta(G)$ is the minimun degree of G.*

By this proposition we can conclude that if a graph is k-connected then it is also k-edge connected but the converse fail.

A graph without cycles is called *acyclic*. An acyclic graph is usually called a *forest*, a connected forest is called a *tree*. Every non trivial tree has always at least one vertex of degree 1, these vertices are called *leaves*.

Proposition 1.0.4. *Given a connected graph T, the following are equivalent:*

(i) *T is a tree;*

(ii) *Any two vertices in T are linked by a unique path in T;*

(iii) *T is minimally connected, that is T is connected but $T-e$ is disconnected for every $e \in E$;*

(iv) *T is maximally acyclic, i.e. T contains no cycle but $T+xy$ does, for very two non-adjacent vertices x,$y \in T$.*

An immediate consequence of this proposition is the following

Corollary 1.0.5. *Every connected graph contains a spanning tree.*

Corollary 1.0.6. *A connected graph with n vertices is a tree if and only if it has $n-1$ edges.*

For our purposes, it is necessary to define a very important operation on a graph: the *edge contraction*. Let $e = (u, v)$ be an edge in G, by G/e we denote the graph obtained from G by *contracting* the edge e into a new vertex v_e, which becomes adjacent to all the previous neighbours of u and v. So $G/e = G' = (V', E')$ with

$$V' := (V \setminus \{u, v\}) \cup \{v_e\}$$

and

$$E' := \{f \in E \mid f \cap e = \emptyset\} \cup \{v_e w \mid uw \in E \text{ or } vw \in E\},$$

thus we just delete e, identify u and v and then create an edge from v_e to every vertices that were adjacent to u or v and we don't mind if any loops or multiple edges arise.

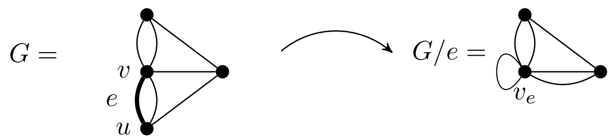

Figure 1.1: Contraction of the edge e

If G is a graph with n components, we define its *first Betti number* as

$$b_1(G) := n - |V(G)| + |E(G)|.$$

A subgraph Δ of G is a *cycle* if it is connected, free from separating edges and $b_1(\Delta) = 1$. If S is a subset of $E(G)$ we define $G(S)$ as the graph obtained from G by contracting all the edges in $G - S$. So there's a (surjective and continous) contraction map $\sigma_S : G \longrightarrow G(S)$ sending to a point every component of $G - S$. Thus $E(G(S)) = S$ and $|V(G(S))| = m$ where m is the number of components of $G - S$. If $S = E$ then $G(S) = G$ and $G(\emptyset)$ is a graph with n isolated vertices, one for every component of G.

Lemma 1.0.7. *G is connected if and only if $G(S)$ is connected.*

There's an useful formula that involves the first Betti number of a graph:

Lemma 1.0.8. *If G is a graph and $S \subseteq E(G)$ then*

$$b_1(G) = b_1(G - S) + b_1(G(S)).$$

Definition 1.0.9. Let G and G' be two graphs. We say that a bijection between their edges $\epsilon : E(G) \longrightarrow E(G')$ is *cyclic* if it induces a bijection between the cycles of G and the cycles of G'.
We say that G and G' are *cyclically equivalent* or *2-isomorphic*, $G \equiv_{cyc} G'$, if there exists a cyclic bijection between their edges.
The cyclic equivalence class of a graph G will be denoted $[G]_{cyc}$.

To describe the cyclic equivalence class of a graph we can consider the following theorem of H. Whitney. A proof can be found in [Whit].

Theorem 1.0.10. *Two graphs, G and G', are cyclically equivalent if and only if they can be obtained from one another via iterated applications of the following two moves:*

(1) Vertex gluing

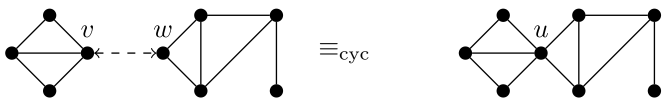

Figure 1.2: Vertex gluing. (The dashed arrow means identification)

(2) Twisting at a separating pair of vertices

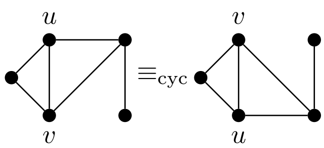

Figure 1.3: Twisting at the separating pair of vertices $\{u, v\}$.

We have an easy consequence of this theorem.

Corollary 1.0.11. *Let G, G' be two graphs such that G is 3-connected. Then $G \equiv_{cyc} G'$ if and only if $G \cong G'$.*

We are now going to define a very special class of subsets of the edge set of a graph, the *C1-sets*.

Definition 1.0.12. Let G be a connected graph such that $E(G)_{\text{sep}} = \emptyset$. Let $S \subseteq E(G)$; we say that S is a *C1-set* of G if

- $G(S)$ is a cycle
- $G - S$ has no separating edge.

In general, if $\widetilde{G} = G - E(G)_{sep}$, we say that S is a C1-set of G if S is a C1-set of a component of $\widetilde{G}$.
The set of all C1-sets of a graph will be denoted $Set^1 G$.

It can be very hard to tell whether a subset of edges of a graph is a C1-set or not. The definition by itself doesn't give us a way to identify the C1-sets of a graph without contracting edges of transforming the original graph.
The following lemma gives us an easier way to find which edges belong to some C1-set and how the elements of these subsets relate to each other.

Lemma 1.0.13. *Let G be a graph and $e, e' \in E(G)$. Then*

(i) Every C1-set S of G satisfies $S \cap E(G)_{sep} = \emptyset$.

(ii) Every non-separating edge e of G is contained in a unique C1-set, S_e. If $E(G)_{sep} = \emptyset$, then $S_e = E(G - e)_{sep} \cup e$.

(iii) e and e' belong to the same C1-set if and only if they belong to the same cycles.

(iv) If G is connected and $e, e' \notin E(G)_{sep}$, then e and e' belong to the same C1-set if and only if (e, e') is a separating pair of edges.

Remark 1.0.14. By the lemma above it is to see that the set of edges of a cycle of a graph is a disjoint union of C1-sets. So we can define

$$\mathrm{Set}^1_\Delta := \{S \in \mathrm{Set}^1 G : S \subset E(\Delta)\}$$

and we have

$$E(\Delta) = \coprod_{S \in \mathrm{Set}^1_\Delta} S.$$

Lemma 1.0.13 yields a very useful corollary that allows us to characterize a class of graphs that play an importan role: the 3-edge connected graphs.

Corollary 1.0.15. *A graph G is 3-edge connected if and only if there's a bijection $E(G) \longrightarrow Set^1 G$ mapping $e \in E(G)$ to $\{e\} \in Set^1 G$*

Corollary 1.0.16. *Let G and G' be two ciclically equivalent graphs, then $|E(G)_{sep}| = |E(G')_{sep}|$. Let $\epsilon : E(G) \longrightarrow E(G')$ be a cyclic bijection, then ϵ induces a bijection*

$$\beta_\epsilon : Set^1 G \longrightarrow Set^1 G'$$

$$S_e \longrightarrow S_{\epsilon(e)}$$

such that $|S| = |\beta_\epsilon(S)|$ for every $S \in Set^1 G$.

The cyclic equivalence class of a graph is a very useful tool for our purposes and that's why we want to find an easier way to tell wheter two graphs are cyclically equivalent or not. So we are now going to define two types of edge contraction that come in handy.

Definition 1.0.17. Given a connected graph G, we define two types of edge contractions, (A) and (B) as follows:

(A) Contraction of a separating edge

(B) Contraction of an edge of a separing pair of edges.

The *2-edge connectivization* of G is the 2-edge connected graph G^2, obtained from G by iterating the operation (A) for all the separating edges of G.
The *3-edge connectivization* of G is the 3-edge connected graph G^3, obtained from G^2 by iterating the operation (B) for all the separating pair of edges of G^2 until there are no separating pairs left.
If G is not connected we define G^2 and G^3 as the 2 and 3-edge connectivization of its components.

As before there's a (surjective) *contraction map* $\sigma : G \to G^2 \to G^3$ obtained by composing the contaction maps defining G^2 and G^3.

Remark 1.0.18. The 2-edge connectivization of a graph is always uniquely determined while the 3-edge connectivization is not.

Although the 3-edge connectivization of a graph is not uniquely determined, we have that any two such connectivizations are cyclically equivalent as we'll prove soon. The connectivization of a graph mirrors many important properties of the starting graph, many of them related to the cyclic equivalence class of G, so, wherever possible, we will look at G^2 or G^3 rather than G.

Lemma 1.0.19. *Let G be a graph.*

(i) *We have* $b_1(G^3) = b_1(G^2) = b_1(G)$.

(ii) *There are two canonical bijections*

$$Set^1 G^3 \longleftrightarrow E(G^3) \longleftrightarrow Set^1 G.$$

(iii) *Two 3-edge connectivizations of* G *are cyclically equivalent.*

(iv) $G^2 \equiv_{cyc} G - E(G)_{sep}$.

Proposition 1.0.20. *Let* G *and* G' *be two graphs.*

(i) *Assume* $G^2 \equiv_{cyc} G'^2$. *Then* $G \equiv_{cyc} G'$ *if and only if* $|E(G)_{sep}| = |E(G')_{sep}|$.

(ii) *Assume* $G^3 \equiv_{cyc} G'^3$ *and* $E(G)_{sep} = |E(G')_{sep}| = \emptyset$. *Then* $G \equiv_{cyc} G'$ *if and only if the natural bijection*

$$\beta : Set^1 G \xrightarrow{\psi} E(G^3) \xrightarrow{\epsilon^3} E(G'^3) \xrightarrow{(\psi')^{-1}} Set^1 G'$$

satisfies $|S| = |\beta(S)|$, *where* ψ *and* ψ' *are the bijections defined in Lemma 1.0.19,* ϵ^3 *is a cyclic bijection and* $S \in Set^1 G$.

Remark 1.0.21. By this result, and by the bijection between the C1-sets of G^3 and G, we can conclude that the cyclic equivalence class of the 3-edge connectivization of a graph depends solely on $[G]_{\text{cyc}}$. Furthermore, suppose G connected, since any two 3-edge connectivizations of G are 3-edge connected and 2-isomorphic then we will refer to $[G^3]_{\text{cyc}}$ as the *3-edge connected class of* G.

If we are interested in giving a *direction* to some edge, that is, we would like to say that an edge *starts from* x and *ends in* y, then we can modify our definition by saying that a *directed graph* (or *digraph*) is a graph such that its edge set is a multiset whose elements are *ordered pair* of vertices. If $e = (x, y) \in E(G)$ then we say that e is an edge *from* x *to* y and they are its *source* and *target* point. The *orientation* of a graph is obtained by assigning a direction to every edge. Any directed graph constructed this way is called an *oriented graph*. We say that a cycle is an *oriented cycle* if it has all the edges oriented in the same direction. A path, P, is *oriented* if all its edges are oriented in the same direction, i.e., if $P = v_1 v_2 \dots v_k$ then every edge in P goes from v_i to v_{i+1} and we say that the path goes *from* v_1 *to* v_k. A *directed acyclic graph* is an oriented graph with no oriented cycles.

Definition 1.0.22. If G is connected, we say that an orientation on G is *totally cyclic* if there exists no proper non-empty subset $W \subset V(G)$ such that the edges between W and its complement $V(G) \smallsetminus W$ go all in the same direction, that is, either all from W to $V(G) \smallsetminus W$ or all in the opposite direction.
If G is not connected that we say that G is totally cyclic if the orientation induced on each component of G is totally cyclic.

Theorem 1.0.23 (Robbins's Theorem)**.** *A connected graph G admits a totally cyclic orientations if and only if $E(G)_{sep} = \emptyset$.*

So every 2-edge connected graph admits a totally cyclic orientation.
We want to study now the following useful lemma.

Lemma 1.0.24. *Let G be an oriented graph with $E(G)_{sep} = \emptyset$. The following are equivalent:*

(a) the orientation of G is totally cyclic;

(b) for any distinct u, $w \in V(G)$ belonging to the same component of G, there exists a path oriented from w to v;

(c) $H_1(G, \mathbb{Z})$ has a basis of cyclically oriented cycles;

(d) every edge $e \in E(G)$ is contained in a cyclically oriented cycle.

Chapter 2

Torelli theorem for graphs

In this chapter we shall use the results of Chapter 1 to prove the Torelli Theorem for graphs. To prove the theorem, we need to associate to a graph two important objects, the Albanese Torus and the Delaunay decomposition. We will follow the work of [OS], [BHN] and [Al]. We start this chapter by recalling the steps needed in order to costruct the first homology group of a graph.
Given an oriented graph G, we can define two functions from the edge set to the vertex set of G which send every edge to its source and target point. So if $e = uv$ is an oriented edge from u to v, we define:

$$\begin{aligned} s : E(G) &\longrightarrow V(G), & t : E(G) &\longrightarrow V(G) \\ e &\longmapsto u & e &\longmapsto v. \end{aligned}$$

Definition 2.0.25. If $G = (V, E)$ is an oriented graph and A is an abelian group, we define:

- The *space of 0-chains of G with values in A* as

$$C_0(G, A) := \bigoplus_{v \in V(G)} A \cdot v$$

- The *space of 1-chains of G with values in A* as

$$C_1(G, A) := \bigoplus_{e \in E(G)} A \cdot e$$

Remember that all the sums are purely *formal*.

We have thus defined our spaces of chains. The second and last thing that we need is a *boundary map* that goes from C_1 to C_0 and we define it as follows:

$$\partial : C_1(G, A) \longrightarrow C_0(G, A)$$

$$e \longmapsto t(e) - s(e)$$

Definition 2.0.26. The *first homology group of a graph G with values in an abelian group A* is $H_1(G, A) := \ker \partial$.

Remark 2.0.27. Notice that $H_1(G, A)$ is independent from the choice of the orientation of G.

We need to set up some notation

Notation 2.0.28. For any edge $e \in E(G)$, we denote by $e^* \in C_1(G, \mathbb{R})^*$ the functional on $C_1(G, \mathbb{R})$ defined as, for $e' \in E(G)$,

$$e^*(e') = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{otherwise.} \end{cases}$$

We will abuse notation by calling $e^* \in H_1(G, \mathbb{R})^*$ also the restriction of e^* to $H_1(G, \mathbb{R})$ and we will keep denoting this scalar product on $H_1(G, \mathbb{R})$ as $(\ , \ )$.

Remark 2.0.29. $e \in E(G)_{\text{sep}}$ if and only if the restriction of e^* to $H_1(G, \mathbb{R})$ is zero since bridges don't lie on any cycle.

Usually, to give rise to a torus, one can define a discrete *lattice* into some bigger space and then perform a quotient. In our case, we consider the lattice $H_1(G, \mathbb{Z})$ inside $H_1(G, \mathbb{R})$ and, passing to the quotient, we obtain a torus defined as $H_1(G, \mathbb{Z})/H_1(G, \mathbb{R})$.

Definition 2.0.30. The *Albanese torus*, *Alb(G)*, of a graph G is

$$\mathrm{Alb}(G) := (H_1(G, \mathbb{R})/H_1(G, \mathbb{Z}); (\ , \ ))$$

with the flat metric derived from the scalar product defined before.

Remark 2.0.31. In the previous chapter we proved that every graph contains a spanning tree, T, and it is well known that such a tree can be retracted to a point. The T-*chords* (edges that are not contained in T that joins two distinct vertices of the graph) become cycles in the graph obtained contracting every edge of T, and so the first homology group is entirely determined by the number of these T-chords. So if we consider a spanning tree $T \subset G$ such that $\{e_1, \dots, e_k\}$ are the T-chords in G, then $H_1(G, A)$ is free abelian on k generators.

It is clear that if we consider a graph G, with a spanning tree T, and its 2-edge connectivization G^2, the latter is obtained by performing an edge contraction for every bridge of G, that are of course contained in T. From all of this follows immediately that $H_1(G, A) \cong H_1(G^2, A)$. If we consider now the inclusion $H_1(G, A) \hookrightarrow C_1(G, A)$ and notice that the edges of G^2 can be naturally identified as $E(G^2) = E(G) \smallsetminus E(G)_{\text{sep}} \subset E(G)$ then the diagram below follows.

$$\begin{tikzcd} H_1(G^2, \mathbb{Z}) \arrow[r, "\tilde{j}", "\cong"'] \arrow[d, hook] & H_1(G, \mathbb{Z}) \arrow[d, hook] \\ C_1(G^2, \mathbb{Z}) \arrow[r, hook, "j"] & C_1(G, \mathbb{Z}). \end{tikzcd}$$

The vertical maps are the inclusion, j is the isclusion induced by the edge identification above and $\tilde{j}$ is the restriction induced by j.

Proposition 2.0.32. *Let G be a graph.*

(i) Alb(G) depends only on $[G]_{cyc}$

(ii) Alb(G) = Alb(G^2)

Corollary 2.0.33. *If $G^2 \equiv_{cyc} G'^2$ then Alb(G) = Alb(G').*

Definition 2.0.34. Consider the finite dimensional real vector space $H_1(G, \mathbb{R})$ and the $\mathbb{Z}$-lattice $H_1(G, \mathbb{Z})$ inside it. The scalar product induced by (,) on $C_1(G, \mathbb{R})$ coincide with the usual euclidean scalar product. We denote the *norm* of an element x by $||x|| := (x, x)^{\frac{1}{2}}$.
Let $\alpha \in H_1(G, \mathbb{R})$. An element x of the lattice $H_1(G, \mathbb{Z})$ is called α*-nearest* if

$$||x - \alpha|| = \min\{||y - \alpha|| \ : \ y \in H_1(G, \mathbb{Z})\}.$$

For a fixed $\alpha \in H_1(G, \mathbb{R})$ we consider the convex hull

$$D(\alpha) := \langle x_0, \dots, x_k \rangle = \left\{ \sum a_i x_i \geq 0, \ \sum a_i = 1 \right\}$$

where $\{x_0, \dots, x_k\}$ is the set of the α-nearest elements of $H_1(G, \mathbb{Z})$.
A set of the form $D(\alpha)$ is called a *Delaunay cell* (or *Delaunay polyhedron*) and we denote by $\text{Del}(G)$ the set of polyhedra in $H_1(G, \mathbb{R})$ of the form $D = D(\alpha)$ and it is the *Delaunay decomposition of G*.
If G and G' are two graphs we say that $\text{Del}(G) \cong \text{Del}(G')$ if there exists a linear isomorphism $H_1(G, \mathbb{R}) \longrightarrow H_1(G', \mathbb{R})$ sending the lattice $H_1(G, \mathbb{Z})$ into the lattice $H_1(G', \mathbb{Z})$ and mapping every Delaunay cell in $\text{Del}(G)$ into the Delaunay cells of $\text{Del}(G')$.

Remark 2.0.35. What we said in Remark 2.0.31 and in Proposition 2.0.32 about the Albanese torus applies also in this case. So it is clear that $\mathrm{Del}(G)$ is uniquely determined by the inclusion of the first homology group into the space of 1-chains, and that, again $\mathrm{Del}(G) \cong \mathrm{Del}(G^2)$ (check Remark 2.0.29).

Now recall the notation for $S \in \mathrm{Set}^1 G$ used in Lemma 1.0.19, the following result is a corollary of Lemma 1.0.24.

Corollary 2.0.36. *Let G be a graph, and fix an orientation inducing a totally cyclic orientation on $G - E(G)_{sep}$. Then the following facts hold.*

(1) For every $c \in H_1(G, \mathbb{Z})$ we have

$$c = \sum_{S \in Set^1 G} r_S(c) \sum_{i=1}^{\#S} e_{S,i}, \quad r_S(c) \in \mathbb{Z}.$$

(2) Let e_1, $e_2 \in E(G) \smallsetminus E(G)_{sep}$. There exists $u \in \mathbb{R}$ such that $e_1^* = ue_2^*$ on $H_1(G, \mathbb{R})$ if and only if e_1 and e_2 belong to the same C1-set of G; moreover, in this case, $u = 1$.

To prove the next proposition we need the following result from [Al]:

Theorem 2.0.37. *The 0-skeleton of the hyperplane arrangment $\{e^* = n$ with $n \in \mathbb{Z}\}$ is the lattice $H_1(G, \mathbb{Z})$ itself.*

Proposition 2.0.38. *Let G and G' be two graphs.*

(i) Del(G) depends only on $[G]_{cyc}$.

(ii) Del(G) $\cong$ Del(G^3) for any choice of G^3.

(iii) Del(G) $\cong$ Del(G') if and only if $G^3 \equiv_{cyc} G'^3$.

We are now ready to state and prove the Torelli theorem for graphs that gives us a necessary and sufficient condition for two graphs to have isomophic associated Albanese torus.

Theorem 2.0.39 (Torelli theorem for graphs)**.** *Let G and G' be two graphs. Then Alb(G) $\cong$ Alb(G') if and only if $G^2 \equiv_{cyc} G'^2$.*

A straightforward corrolary of the Torelli theorem is the following.

Corollary 2.0.40. *Let G be a 3-connected graph and let G' be a graph with no vertex of valency 1. Then Alb(G) $\cong$ Alb(G') if and only if $G \cong G'$*

Chapter 3

Tropical curves and metric graphs

In this chapter we introduce the reader to tropical geometry, defining firstly the tropical semifield and the tropical operations. After defining abstract tropical curves we will show the correspondence between these curves and metric graphs. If we assign a metric to a graph, then we need to rephrase the results of Chapter 2 to let them work in the metric case. Since the theoretical results are quite similar to those of the non-metric case, we will omit them, sending, the reader to the whole thesis if interested. However, we propose a brief introduction to tropical geometry.

Tropical algebra is defined on the set of *tropical numbers* $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ endowed with the two *tropical operations* (we will use quotation marks to make a distinction between usual operations and tropical operations):

- *Tropical sum*: "$x + y$" $:= \max\{x, y\}$;
- *Tropical product*: "$x \cdot y$" = "xy" $:= x + y$.

Definition 3.0.41. We can now define the *tropical semifield* as

$$(\mathbb{T}, \text{“}+\text{”}, \text{“}\cdot\text{”})$$

that is a commutative semiring, i.e. a set equipped with commutative and associative operations of addition and multiplication so that the distribution law holds while the addition and multiplication operations have neutral elements, such that the non-zero elements $\mathbb{T}^{\times}$ form a group with respect to multiplication.

We can define a topology on the semifield $\mathbb{T}$ by identifying it with the half-open interval $[-\infty, +\infty)$. This topology is generated by the sets $\{x \in \mathbb{T} \mid x > a\}$ and $\{x \in \mathbb{T} \mid x < b\}$ for $a, b \in \mathbb{T}^\times = (-\infty, +\infty)$. From an algebraic point of view, this two sets can be described as $\{x \in \mathbb{T} \smallsetminus \{a\} \mid \text{“}x + a\text{”} = x\}$ and $\{x \in \mathbb{T} \smallsetminus \{b\} \mid \text{“}x + b\text{”} = b\}$.

Definition 3.0.42. The *tropical affine n-space* is the topological space defined as

$$\mathbb{T}^n = [-\infty, +\infty)^n$$

and the so called *n-torus* is

$$(\mathbb{T}^\times)^n = (-\infty, +\infty)^n = \mathbb{R}^n \subsetneq \mathbb{T}^n.$$

Once we have defined the tropical operations we can define a *tropical polynomial* as $P(x) = \text{“}\sum_{i=0}^{d} a_i x^i\text{”}$ with $a_i \in \mathbb{T}$, hence $P(x) = \max_{i=0}^{d}\{a_i + ix\}$. The first thing that we might want to ask is *what is a tropical root*? In the real case we know that a root would be a value x_0 such that $P(x_0) = 0$, hence, if we bring the definition to the tropical case we should look for those x_0 such that $P(x_0) = -\infty$. If a_0 is the constant term of $P(x)$ then $P(x) \geq a_0$ since $\max\{a_0, x\} \geq a_0$ for every $x \in \mathbb{T}$. Therefore, if $a_0 \neq -\infty$ then our polynomial would have no roots.
This definition is clearly unsatisfying but, if we recall the fact that we call x_0 a root of $P(x)$ if there exists a polynomial $Q(x)$ such that $P(x) = (x - x_0)Q(x)$, then we have a correct definition.

Definition 3.0.43. The *tropical roots* of a tropical polynomial $P(x) = \text{“}\sum_{i=0}^{d} a_i x^i\text{”} = \max_{i=0}^{d}\{a_i + ix\}$ are exactly the values x_0 such that there exist $i \neq j$ such that $P(x_0) = a_i + ix_0 = a_j + jx_0$. So the roots are those values for which the maximum is reached more than once. The *degree* of the root is equal to $\max\{|i - j|\}$ for all possible i and j that realize the maximum. Equivalently, x_0 is a tropical root of order k for $P(x)$ if there exists a polynomial $Q(x)$ such that $P(x) = \text{“}(x + x_0)^k Q(x)\text{”}$.

Notice that we can look at $P(x)$ as a function

$$P : \mathbb{T} \longrightarrow \mathbb{T}$$

$$x \longmapsto P(x).$$

Since we want to study tropical curves, we need a definition for a two variables polynomial.

Definition 3.0.44. Let $P(x,y) = \text{“}\sum_{i,j} a_{i,j}x^i y^j\text{”} = \max\{a_{i,j} + ix + jy\}$ be a tropical polynomial. The *tropical curve* C, defined by $P(x,y)$ is the set of points (x_0, y_0) of $\mathbb{R}^2$ such that there exist $(i,j) \neq (k,l)$ verifying $P(x_0,y_0) = a_{i,j} + ix_0 + jy_0 = a_{k,l} + kx_0 + ly_0$. In other words, C is the locus where the maximum is reached by more than one of the “monomials” of P.

So we have that a tropical curve is piecewise-linear. More precisely, is the union of either straight rays or segments, both with rational slope and the points of intersection of these are called *vertices* (or *nodes*) while the linear pieces are called *edges*, a point lying on an edge is called *inner* and for every vertex v we define its *valency* as the number of edges that intersect at v.

Definition 3.0.45. The *Tropical projective space* $\mathbb{TP}^n$ consists of the classes of $(n+1)$-tuple of tropical numbers, not all of them equal to $-\infty$, with respect to the equivalence relation

$$(x_0 : \cdots : x_n) \sim (y_0 : \cdots : y_n) :\iff \exists\, \lambda \in \mathbb{T}^\times \text{ such that } x_i = \lambda + y_i \text{ for } i = 0, \ldots, n.$$

Clearly if we assign to the n-tuple $(x_1, \ldots, x_n) \in \mathbb{R}^n = (\mathbb{T}^\times)^n$ the $n+1$-tuple $(0 : x_1 : \cdots : x_n) \in \mathbb{TP}^n$ we have an embedding $\iota_n : \mathbb{R}^n \hookrightarrow \mathbb{TP}^n$.

Let $V_n \subset \mathbb{T}^n$ be subspace generated by

$$e_1 = (-\infty, 0, \ldots, 0),\ e_2 = (0, -\infty, 0, \ldots, 0), \ldots,\ e_n = (0, \ldots, 0, -\infty),$$

that is $V_n = \{v \in \mathbb{T}^n \ : \ v = \text{“}\sum_{j=1}^n c_j e_j\text{”} \text{ with } c_j \in \mathbb{T}\}$. Consider now the equivalence classes of the elements of V_n, $\mathbb{P}(V_n)$, given by

$$v \sim v' \ :\iff \ v = \text{“}\lambda v'\text{”},\ \exists\, \lambda \in \mathbb{T}^\times.$$

$\mathbb{P}(V_n)$ is called the *projectivization of* V_n and we set $\mathbb{P}(V_n) = \Gamma_n$. A different, and for us useful alternative definition for tropical curves comes by defining them abstracly as follows.

Definition 3.0.46. A $\mathbb{Z}$-*affine map* is an affine-linear map $A : \mathbb{R}^k \longrightarrow \mathbb{R}^{k'}$, where A decomposes into a linear map $L_A : \mathbb{R}^k \longrightarrow \mathbb{R}^{k'}$ and a translation, such that its linear part, L_A, in defined over $\mathbb{Z}$, that is, L_A is a matrix with integer entries.

Now let C be a connected topological space homeomorphic to a locally finite 1-dimensional simplicial complex (i.e. C is homeomorphic to a graph). A *complete tropical structure* on C is an open covering $\{U_\alpha\}$ of C with a collection of embeddings

$$\phi_\alpha : U_\alpha \hookrightarrow \mathbb{R}^{k_\alpha - 1}$$

called the *charts*, with $k_\alpha \in \mathbb{N}$, such that for every α and β the following conditions hold:

- $\phi(U_\alpha) \subset \Gamma_{k_\alpha}$.
- If $U' \subset U_\alpha$ is an open subset then $\phi_\alpha(U')$ is open in $\Gamma_{k_\alpha} \subset \mathbb{TP}^{k_\alpha - 1}$.
- Whenever $U_\alpha \cap U_\beta \neq \emptyset$ we can define an overlap map $\iota^{-1}_{k_\alpha - 1} \circ \phi_\alpha \circ \phi^{-1}_\beta \circ \iota_{k_\beta - 1}$ such that it is a restriction of a $\mathbb{Z}$-affine linear map $\mathbb{R}^{k_\beta - 1} \longrightarrow \mathbb{R}^{k_\alpha - 1}$. The map ι_n is the embedding of $\mathbb{R}^n$ into $\mathbb{TP}^n$ defined before.
- If $S \subset U_\alpha$ is a closed set in C then $f(S) \cap \iota_{k_\alpha - 1}(\mathbb{R}^{k_\alpha - 1})$ is a closed set in $\iota_{k_\alpha - 1}(\mathbb{R}^{k_\alpha - 1})$.

The space C equipped with a complete tropical structure is called a *tropical curve.*

We are interested on *compact* tropical curves and the compactness of a tropical curve fails when it has some unbounded edges. We can overcome this problem by adding a point at infinity for each open end creating a 1-valent vertex.

Definition 3.0.47. A *metric graph* (G, l) is a finite graph G endowed with a function

$$l : E(G) \longrightarrow \mathbb{R}_{>0}$$

called the *lenght function.*

So a metric graph is just a usual graph with a length assigned to every edge.

Remark 3.0.48. A metric graph is a complete space.

Thanks to [MZ] we can state the following proposition that gives us a way to look at tropical curves as metric graph and viceversa.

Proposition 3.0.49. *There is a natural 1-1 correspondence between compact tropical curves and metric graphs.*

Definition 3.0.50. We say that two tropical curves, C and C', are *tropically equivalent* if one can be obtained from the other by suppressing or adding a vertex of degree 1 together with its adjacent edge, or a 2-valent vertex over an edge as shown in figure.

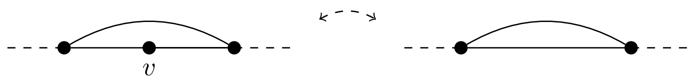

Figure 3.1: Suppressing/adding the 2-valent vertex v

Remark 3.0.51. Since in [MZ] the length function assumes the value $+\infty$ on the leaves of G, to avoid confusion among definitions, we shall assume that our graphs have minimum degree 2; moreover, since we shall consider compact tropical curves up to tropical equivalence, then we can assume that the associated metric graphs have minimum degree 3, as we can suppress every vertex with degree smaller than three. So we can choose a unique (up to isomorphism) representative graph for each tropical equivalence class of tropical curves.
We will use the graph theoretic terminology also for tropical curves. In particular we shall say that C is k-connected if so is G.

If we assign a length function to a graph then we need to rephrase the definitions given in the previous chapters. More precisely we have to give the analogous definitions for the Albanese torus, cyclic equivalence and the 3-edge connectivization of a metric graph. Clearly, most of the previous results apply also on these new graphs.

Definition 3.0.52. Let (G,l) and (G',l') be two metric graphs. We say that (G,l) and (G',l') are *cyclically equivalent*, and we write $(G,l) \equiv_{\text{cyc}} (G',l')$, if there exists a cyclic bijection $\epsilon : E(G) \longrightarrow E(G')$ such that $l(e) = l'(\epsilon(e))$ for all $e \in E(G)$. The cyclic equivalence class of (G,l) will be denoted by $[(G,l)]_{cyc}$.

Definition 3.0.53. A *3-edge connectivization* of a metric graph (G,l) is a metric graph (G^3,l^3) where G^3 is the 3-edge connectivization of G, and l^3 is the length function defined as follows,

$$l^3(e_S) = \sum_{e\in\psi^{-1}(e_S)} l(e) = \sum_{e\in S} l(e)$$

where $S \in \text{Set}^1 G$ and ψ is the natural bijection defined in 1.0.19

$$\psi : \text{Set}^1 G \longrightarrow E(G^3)$$

$$S \longmapsto e(S).$$

Notice that by Lemma 1.0.19 part (iii) we still have a cyclic equivalence between any 3-edge connectivizations of a metric graph.

Definition 3.0.54. Given a metric graph (G, l), we define the scalar product $(,)_l$ on $C_1(G, \mathbb{R})$ as follows

$$(e, e')_l = \begin{cases} l(e) & \text{if } e = e', \\ 0 & \text{otherwise.} \end{cases}$$

The *Albanese torus* $\mathrm{Alb}(G, l)$ of the metric graph (G, l) is

$$\mathrm{Alb}(G, l) := (H_1(G, \mathbb{R})/H_1(G, \mathbb{Z}); (,)_l)$$

with the metric derived from the scalar product $(,)_l$.

Definition 3.0.55. We say that a tropical curve C is *irreducible* if it is connected as a topological space. The *genus* of an irreducible tropical curve is $g = \mathrm{dim} H_1(C, \mathbb{R}) = b_1(C)$. We define, according to [MZ], the *Jacobian variety* of a tropical curve C to be

$$\mathrm{Jac}(C) := \Omega(C)^*/H_1(C, \mathbb{Z}) \cong \mathbb{R}^g/\Lambda$$

where g is the genus of the tropical curve, $\Omega(C)$ is the space of global 1-forms on C, $\Omega(C)^*$ is the vector space of $\mathbb{R}$-valued linear functionals on $\Omega(C)$ and Λ is the lattice in $\Omega(C)^*$ obtained by integrating over the cycles of $H_1(C, \mathbb{Z})$. Now consider the positive definite symmetric bilinear form $Q : \Omega(C)^* \longrightarrow \mathbb{R}$, defined by the metric of C by $Q(\gamma, \gamma) := l(\gamma)$ on simple cycles γ (i.e. γ has no self intersections), and the *tropical theta function* given by

$$\Theta(x) := \max_{\lambda \in \Lambda} \{Q(\lambda, x) - \frac{1}{2} Q(\lambda, \lambda)\}, \quad x \in \Omega(C)^*.$$

We have that $\Theta(-x) = \Theta(x)$ and, for every $\mu \in \Lambda$, $\Theta(x + \mu) = \Theta(x) + \frac{1}{2} Q(\mu, \mu)$. The locus in which the maximum is attained for more than one λ, denoted as $\Theta_C \subset \mathrm{Jac}(C)$, is a divisor on $\mathrm{Jac}(C)$ and it is called *theta divisor*.

Definition 3.0.56. We shall denote $(\mathrm{Jac}(C), \Theta_C)$ the *principally polarized Jacobian* of C, where the principal polarization is given by the theta divisor Θ_C.

Remark 3.0.57. Giving a principal polarization by the theta divisor on an abelian variety is equivalent to giving a metric. By [MZ] we can naturally identify $(\mathrm{Jac}(C), \Theta_C)$ with $\mathrm{Alb}(G, l)$.

Definition 3.0.58. The *Delaunay decomposition* of a metric graph, $\mathrm{Del}(G,l)$, is the Delaunay decomposition (cf. Definition 2.0.34) of G associated to the scalar product $(\ , \ )_l$ defined on $H_1(G,\mathbb{R})$ with respect to the lattice $H_1(G,\mathbb{Z})$.

We are now going to state and prove a "metric" analogue of the results needed in order to prove the metric and tropical version of the Torelli theorem. As we expect, the properties of $\mathrm{Alb}(G,l)$ and $\mathrm{Del}(G,l)$ mirror those of Alb(G) and $\mathrm{Del}(G)$.
In the subsequent Proposition, the reader will surely notice that, differently from Proposition 2.0.32, we will state an isomorphism between the Albanese torus of a metric graph and that of its 3-edge connectivization (instead of the 2-edge connectivization). This occours because the values that the length function assumes on the separating edges of G can be ignored, that is, $[(G^3,l^3)]_{\mathrm{cyc}}$ is completely independent of the separating edges of G and on the value that l takes on them. So $[(G^3,l^3)]_{\mathrm{cyc}}$ is well defined, for a metric graph associated to a tropical curve C, even if l takes value $+\infty$ on the leaves of G. Hence we can call $[(G^3,l^3)]_{\mathrm{cyc}}$ the *3-edge connected class of C*.

Proposition 3.0.59. *Let (G,l) be a metric graph.*

(i) $Alb(G,l)$ depends only on $[(G,l)]_{cyc}$.

(ii) $Alb(G,l) \cong Alb(G^3,l^3)$ for every 3-edge connectivization of (G,l).

Theorem 3.0.60 (Torelli theorem for metric graphs)**.** *Let (G,l) and (G',l') be two metric graphs. Then $Alb(G,l) \cong Alb(G',l')$ if and only if $[(G^3,l^3)]_{cyc} = [(G'^3,l'^3)]_{cyc}$.*

Once we have the metric version of the Torelli theorem we can prove its tropical version.

Theorem 3.0.61 (Torelli theorem for tropical curves)**.** *Let C and C' be compact tropical curves. Then $(Jac(C),\Theta_C) \cong (Jac(C'),\Theta_{C'})$ if and only if C and C' have the same 3-edge connected class.*
Suppose that C is 3-connected. Then $(Jac(C),\Theta_C) \cong (Jac(C'),\Theta_{C'})$ if and only if C and C' are tropically equivalent.

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