Università degli Studi Roma Tre - Corso di Laurea in Matematica

AM3 tutorato 9

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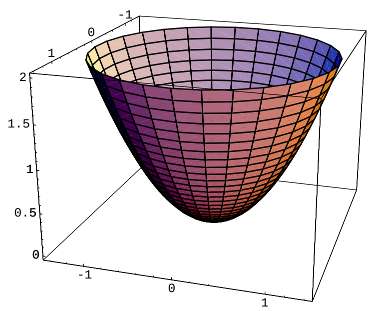
Esercizio 1 Se γ ha equazione polare $\rho(\theta) = \cos\theta + \sin\theta$ con $\theta \in [0, \pi]$ allora una parametrizzazione di γ è $\gamma(t) = (\rho(t)\cos(t), \rho(t)\sin t) = ((\cos t + \sin t)\cos t, (\cos t + \sin t)\sin t)$
$= (\cos^2 t + \cos t \sin t, \cos t \sin t + \sin^2 t)$.
$\dot\gamma\ (t) = (-2\cos t\sin t - \sin^2 t + \cos^2 t, -\sin^2 t + \cos^2 t + 2\sin t\cos t) =$
$= (\cos 2t - \sin 2t, \cos 2t + \sin 2t)$
$|\ \dot\gamma\ (t)| = \sqrt{\cos^2 2t + \sin^2 2t - 2\cos 2t\sin 2t + \cos^2 2t + \sin^2 2t + 2\sin 2t\cos 2t} = \sqrt{2}$
Quindi $l(\gamma) = \int_0^{\pi} |\ \dot\gamma\ (t)|\ dt = \int_0^{\pi} \sqrt{2}dt = \sqrt{2}\pi$

$$\int_\gamma \frac{xy^2}{x^2+y^2}ds = \int_0^{\pi} \frac{\rho(t)^3\cos t\sin^2 t}{\rho(t)^2}|\ \dot\gamma\ (t)|\ dt = \sqrt{2}\int_0^{\pi}(\cos t + \sin t)\cos t\sin^2 t\ dt =$$
$$= \sqrt{2}\int_0^{\pi}\cos^2 t\sin^2 t + \cos t\sin^3 t\ dt = \frac{\sqrt{2}}{4}\int_0^{\pi}\sin^2 2t\ dt + \sqrt{2}\int_0^{\frac{\pi}{2}}\cos t\sin^3 t\ dt +$$
$$+\sqrt{2}\int_{\frac{\pi}{2}}^{\pi}\cos t\sin^3 t\ dt = \frac{\sqrt{2}}{8}\int_0^{2\pi}\sin^2 t\ dt + \sqrt{2}\int_0^1 t^3\ dt - \sqrt{2}\int_0^1 t^3 dt = \frac{\pi}{8}\sqrt{2}$$

Esercizio 2 $\alpha(t) = (\sqrt{1+t^2}, t, \arctan t)\quad t \in [0, 1]$
$\dot\alpha\ (t) = \left(\frac{t}{\sqrt{1+t^2}}, 1, \frac{1}{1+t^2}\right)$

$$|\ \dot\alpha\ (t)| = \sqrt{\frac{t^2}{1+t^2} + 1 + \frac{1}{(1+t^2)^2}} = \frac{\sqrt{t^2(1+t^2) + (1+t^2)^2 + 1}}{(1+t^2)} = \frac{\sqrt{2+3t^2+t^4}}{1+t^2}$$
$$\int_\alpha \frac{y^2\sin z}{\sqrt{2+3y^2+2y^4}}ds\ = \int_0^1 \frac{t^2\sin(\arctan t)}{\sqrt{2+3t^2+t^4}}\frac{\sqrt{2+3t^2+t^4}}{1+t^2}dt = \int_0^1 \frac{t^2\sin(\arctan t)}{1+t^2}dt$$
$$\overset{(y=\arctan t)}{=} \int_0^{\frac{\pi}{4}} \tan^2 y\sin y dy = \int_0^{\frac{\pi}{4}} \frac{(1-\cos^2 y)}{\cos^2 y}\sin y\ dy \overset{(s=\cos y)}{=} \int_{\frac{1}{\sqrt{2}}}^1 \frac{1-s^2}{s^2}ds =$$
$$= \int_{\frac{1}{\sqrt{2}}}^1 \frac{1}{s^2} - 1\ ds = -\frac{1}{s} - s\Big|_{\frac{1}{\sqrt{2}}}^1 = -1 + \sqrt{2} - 1 + \frac{1}{\sqrt{2}} = \frac{3}{2}\sqrt{2} - 2$$

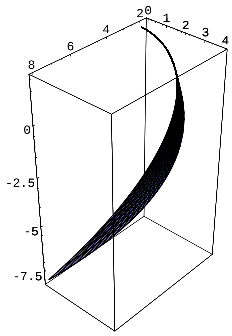
Esercizio 3 $\Phi(u, v) = (u\cos v, u\sin v, u^2)\quad u \in [0, 2\pi]\ v \in [0, \sqrt{2}]$



$\Phi_u = (\cos v, \sin v, 2u)$

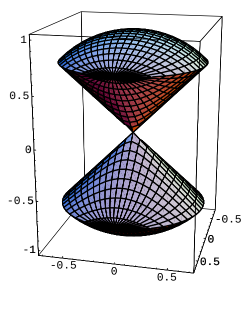
$\Phi_v = (-u \sin v, u \cos v, 0)$
$\Phi_u \wedge \Phi_v = (-2u^2 \cos v, -2u^2 \sin v, u)$
$|\Phi_u \wedge \Phi_v| = \sqrt{4u^4 \cos^2 v + 4u^4 \sin^2 v + u^2} = u\sqrt{1+4u^2}.$
Quindi $Area(\Sigma) = \int_0^{\sqrt{2}} du \int_0^{2\pi} u\sqrt{1+4u^2} = 2\pi \int_0^{\sqrt{2}} du\ u\sqrt{1+4u^2} = \pi \int_0^2 du\ \sqrt{1+4u} =$
$= \pi \frac{1}{6}(1+4u)^{\frac{3}{2}}\Big|_0^2 = \frac{\pi}{6}(27-1) = \frac{13}{3}\pi$

Esercizio 4 $\Phi(u,v) = (u^2+v^2, u^2-v^2, 2uv)$ con $(u,v) \in D = \{(x,y) \in \mathbb{R}^2 \mid \frac{4}{x^2} \leq y \leq 5 - x^2\ ,\ x \geq 0\}$



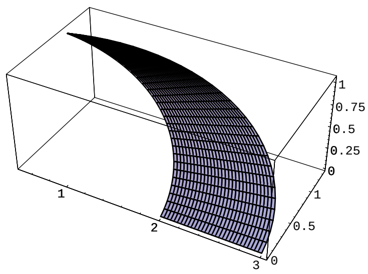
$\Phi_u = (2u, 2u, 2v)$
$\Phi_v = (2v, -2v, 2u)$
$\Phi_u \wedge \Phi_v = (4u^2+4v^2, 4v^2-4u^2, -8vu) = 4(u^2+v^2, v^2-u^2, -2uv)$
$|\Phi_u \wedge \Phi_v| = 4\sqrt{(u^2+v^2)^2+(v^2-u^2)^2+4u^2v^2} = 4\sqrt{2u^2+2v^2+4u^2v^2} = 4\sqrt{2}(u^2+v^2)$
$\int_S \frac{z}{x}\, d\sigma = \int_E \frac{2uv}{u^2+v^2} 4\sqrt{2}(u^2+v^2) du dv = 8\sqrt{2}\int_E uv\ du dv$
$E = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2\ ,\ \frac{4}{x^2} \leq y \leq 5 - x^2\ \}$
$\int_S \frac{z}{x}\, d\sigma = 8\sqrt{2}\int_E uv\ du dv = 8\sqrt{2}\int_1^2 du \int_{\frac{4}{u^2}}^{5-u^2} dv\ uv = 4\sqrt{2}\int_1^2 du\ uv^2\Big|_{\frac{4}{u^2}}^{5-u^2} =$
$= 4\sqrt{2}\int_1^2 du\ u(5-u^2)^2 - u\frac{16}{u^4} = 4\sqrt{2}\int_1^2 du\ u(5-u^2)^2 - \frac{16}{u^3} =$
$= 4\sqrt{2}\left(-\frac{1}{6}(5-u^2)^3 + \frac{8}{u^2}\Big|_1^2\right) = 4\sqrt{2}\left(\frac{32}{3} - \frac{1}{6} + 2 - 8\right) = 4\sqrt{2}\left(\frac{21}{2} - 6\right) = 4\sqrt{2}\,\frac{9}{2} =$
$= 18\sqrt{2}$

Esercizio 5 Dobbiamo calcolare l'area della superficie del bordo dell' insieme



$E = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \leq 1\ ,\ z^2 \geq x^2+y^2\}$.
Per prima cosa notiamo che $Area(\partial E) = 2Area(\partial E')$ dove $E' = E \cap \{z \geq 0\}$.
$\partial E' = \Sigma_1 \cap \Sigma_2$ dove
$\Sigma_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{x^2+y^2}\ ,\ 0 \leq z \leq \frac{1}{\sqrt{2}}\}$ è una porzione della superficie laterale del cono e $\Sigma_2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\ ,\ z \geq \frac{1}{\sqrt{2}}\}$ è una calotta sferica.
Una parametrizzazione di Σ_1 è $\Phi(u,v) = (u\cos v, u\sin v, u)$ con $u \in [0, \frac{1}{\sqrt{2}}]$ e $v \in [0, 2\pi]$ $\quad \Phi_u = (\cos v, \sin v, 1) \quad \Phi_v = (-u\sin v, u\cos v, 0)$.
$\Phi_v \wedge \Phi_v = (-u\cos v, -u\sin v, u)$.
$|\Phi_u \wedge \Phi_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2}u$ quindi
$Area(\Sigma_1) = \displaystyle\int_0^{\frac{1}{\sqrt{2}}} du \int_0^{2\pi} dv\ \sqrt{2}u = 2\pi\sqrt{2}\int_0^{\frac{1}{\sqrt{2}}} u\ du = \frac{\sqrt{2}}{2}\pi$
Una parametrizzazione di Σ_2 è $\Phi(u,v) = (\sin v\cos u, \sin v\sin u, \cos v)$ con $u \in [0, 2\pi]$ e $v \in [0, \frac{\pi}{4}]$ $\quad \Phi_u = (-\sin v\sin u, \sin v\cos u, 0) \quad \Phi_v = (\cos v\cos u, \cos v\sin u, -\sin v)$.
$\Phi_v \wedge \Phi_v = (-\sin^2 v\cos u, -\sin^2 v\sin u, -\sin v\cos v)$.
$|\Phi_u \wedge \Phi_v| = \sqrt{\sin^4 v\cos^2 u + \sin^4 v\sin^2 u + \sin^2 v\cos^2 v} = \sin v$ quindi
$Area(\Sigma_2) = \displaystyle\int_0^{2\pi} du \int_0^{\frac{\pi}{4}} dv\ \sin v = 2\pi\int_0^{\frac{\pi}{4}} \sin v = -2\pi\cos v\Big|_0^{\frac{\pi}{4}} = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right) =$
$= 2\pi - \sqrt{2}\pi$
$Area(\partial E') = Area(\Sigma_1) + Area(\Sigma_2) = 2\pi - \dfrac{\sqrt{2}}{2}\pi$ e quindi
$Area(\partial E) = 2Area(\partial E') = 4\pi - \sqrt{2}\pi$

Esercizio 6 $\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid z = \arctan\frac{y}{x} \mid x^2+y^2 \geq 1\ ,\ x^2+y^2 \leq 2x\ ,\ y \geq 0\ \}$.



Una parametrizzazione di Σ è data dalla funzione $\Phi(u,v) = (u, v, \arctan\frac{v}{u})$ con $(u,v) \in E = \{(x,y) \mid x^2+y^2 \geq 1\ ,\ x^2+y^2 \leq 2x\ ,\ y \geq 0\ \}$
$\Phi_u = (1, 0, -\frac{v}{u^2+v^2}) \quad \Phi_v = (0, 1, \frac{u}{u^2+v^2})$
$\Phi_u \wedge \Phi_v = (\frac{v}{u^2+v^2}, \frac{u}{u^2+v^2}, 1)$
$|\Phi_u \wedge \Phi_v| = \sqrt{1 + \dfrac{u^2+v^2}{(u^2+v^2)^2}} = \sqrt{\dfrac{1+u^2+v^2}{u^2+v^2}}$
$\displaystyle\int_\Sigma \frac{yz}{\sqrt{1+x^2+y^2}} d\sigma = \int_E v\frac{\arctan\frac{v}{u}}{\sqrt{1+u^2+v^2}}\sqrt{\frac{1+u^2+v^2}{u^2+v^2}} dudv = \int_E \frac{v\arctan\frac{v}{u}}{\sqrt{u^2+v^2}} dudv.$
$E = \{(\rho\cos\theta, \rho\sin\theta) \mid \theta \in [0, \dfrac{\pi}{3}]\ ,\ 1 \leq \rho \leq 2\cos\theta\}$ quindi passando in coordinate polari
$\displaystyle\int_E \frac{v\arctan\frac{v}{u}}{\sqrt{u^2+v^2}} dudv = \int_0^{\frac{\pi}{3}} d\theta \int_1^{2\cos\theta} d\rho\ \rho\sin\theta\arctan(\tan\theta) = \int_0^{\frac{\pi}{3}} d\theta \int_1^{2\cos\theta} d\rho\ \rho\theta\sin\theta =$
$\displaystyle = \frac{1}{2}\int_0^{\frac{\pi}{3}} \rho^2\theta\sin\theta\Big|_1^{2\cos\theta} = 2\int_0^{\frac{\pi}{3}} \theta\cos^2\theta\sin\theta\ d\theta - \frac{1}{2}\int_0^{\frac{\pi}{3}} \theta\sin\theta\ d\theta =$

$$= \frac{2}{3}\int_0^{\frac{\pi}{3}} \theta\, 3\cos^2\theta \sin\theta\, d\theta - \frac{1}{2}\left(-\theta\cos\theta\Big|_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \cos\theta\, d\theta\right) =$$
$$= \frac{2}{3}\left(-\theta\cos^3\theta\Big|_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \cos^3\theta\, d\theta\right) - \frac{1}{2}\left(-\frac{\pi}{6} + \sin\theta\Big|_0^{\frac{\pi}{3}}\right) =$$
$$= -\frac{\pi}{36} + \frac{2}{3}\int_0^{\frac{\pi}{3}} \cos^3\theta\, d\theta + \frac{\pi}{12} - \frac{\sqrt{3}}{4} = \frac{\pi}{18} - \frac{\sqrt{3}}{4} + \frac{2}{3}\int_0^{\frac{\pi}{3}} (1-\sin^2 t)\cos t\, dt =$$
$$= \frac{\pi}{18} - \frac{\sqrt{3}}{4} + \frac{2}{3}\int_0^{\frac{\sqrt{3}}{2}} (1-t^2)dt = \frac{\pi}{18} - \frac{\sqrt{3}}{4} + \frac{2}{3}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{8}\right) = \frac{\pi}{18}$$

Esercizio 7 Sia $\omega = (x+y^2)dx + xzdy + xz^3dz$ calcolare $\int_\gamma \omega$ dove $\gamma : [0,\pi] \longrightarrow \mathbb{R}^3$ è la curva $\gamma(t) = (\sin t, \cos t, \sin t)$.
$\dot{\gamma}\,(t) = (\cos t, -\sin t, \cos t)$
Il campo vettoriale associato ad ω è $F(x,y,z) = (x+y^2, xz, xz^3)$. Per definizione si ha che

$$\int_\gamma \omega = \int_0^\pi < F(\gamma(t)), \dot{\gamma}\,(t) > dt = \int_0^\pi < (\sin t + \cos^2 t, \sin^2 t, \sin^4 t), (\cos t, -\sin t, \cos t) >$$
$$= \int_0^\pi \sin t\cos t + \cos^3 t - \sin^3 t + \sin^4 t \cos t\, dt = \frac{1}{2}\int_0^\pi \sin 2t + \int_0^\pi (1-\sin^2 t)\cos t\, dt +$$
$$- \int_0^\pi (1-\cos^2 t)\sin t\, dt + \int_0^\pi \sin^4 t\cos t\, dt = 0 + \int_0^{\frac{\pi}{2}} (1-\sin^2 t)\cos t\, dt + \int_{\frac{\pi}{2}}^\pi (1-\sin^2 t)\cos t +$$
$$- \int_0^\pi (1-\cos^2 t)\sin t\, dt + \frac{1}{5}\sin^5 t\Big|_0^\pi = \int_0^1 (1-t^2)dt - \int_0^1 (1-t^2)dt - \int_{-1}^1 (1-t^2)dt$$
$$= -\int_{-1}^1 (1-t^2)dt = -2\int_0^1 (1-t^2)dr = -2\left(1-\frac{1}{3}\right) = -\frac{4}{3}$$

Esercizio 8 $\omega = \dfrac{e^x}{1+y^2}dx + \left(2y - \dfrac{2e^xy}{(1+y^2)^2}\right)dy$

(a) $\dfrac{d}{dy}\dfrac{e^x}{1+y^2} = -\dfrac{2ye^x}{(1+y^2)^2}$ e $\dfrac{d}{dx}\left(2y - \dfrac{2e^xy}{(1+y^2)^2}\right) = -\dfrac{2e^xy}{(1+y^2)^2}$ quindi ω è una forma differenziale chiusa.

(b) Siccome ω è chiusa e definita su tutto $\mathbb{R}^2$ (che è semplicemente connesso) allora ω è esatta. Cerchiamo un potenziale per ω cioè una funzione $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ tale che $\dfrac{\partial f}{\partial x} = \dfrac{e^x}{1+y^2}$ e $\dfrac{\partial f}{\partial y} = 2y - \dfrac{2e^xy}{(1+y^2)^2}$.
$\dfrac{\partial f}{\partial x} = \dfrac{e^x}{1+y^2} \Longrightarrow f(x,y) = \displaystyle\int \dfrac{e^x}{1+y^2} + c(y) = \dfrac{e^x}{1+y^2} + c(y)$ quindi
$\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial y}\left(\dfrac{e^x}{1+y^2} + c(y)\right) = -\dfrac{2ye^x}{(1+y^2)^2} + \dfrac{\partial c}{\partial y} = 2y - \dfrac{2e^xy}{(1+y^2)^2} \Longrightarrow \dfrac{dc}{dy} = 2y$
$\Longrightarrow c(y) = y^2$ pertanto una primitiva di ω è $f(x,y) = \dfrac{e^x}{1+y^2} + y^2$

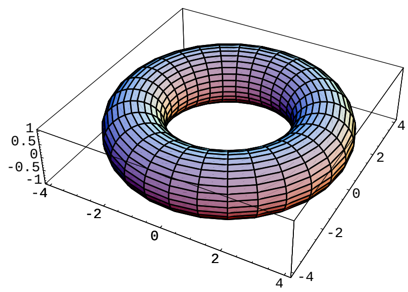
(c) $\gamma(t) = (t\arctan t\ ,\ e^{\frac{\pi}{4}t - t^2})\quad t \in [0, \frac{\pi}{4}]$
Dato che ω è esatta e $\omega = df$ si ha
$\displaystyle\int_\gamma \omega = f(\gamma(\frac{\pi}{4})) - f(\gamma(0)) = f(\frac{\pi}{4}\arctan\frac{\pi}{4}, 1) - f(0,1) = \frac{1}{2}e^{\frac{\pi}{4}\arctan\frac{\pi}{4}} - \frac{1}{2}.$

Esercizio 9 $\omega = \dfrac{2y^3}{(x^2+y^2)^2}dx - \dfrac{2xy^2}{(x^2+y^2)^2}dy$

(a) $\frac{d}{dx} - \frac{2xy^2}{(x^2+y^2)^2} = -2\frac{y^2(x^2+y^2)^2 - 4x^2y^2(x^2+y^2)}{(x^2+y^2)^4} = 2\frac{3x^2y^2 - y^4}{(x^2+y^2)^3}$ e $\frac{d}{dy}\frac{2y^3}{(x^2+y^2)^2} = 2\frac{3y^2(x^2+y^2)^2 - 4y^4(x^2+y^2)}{(x^2+y^2)^4} = 2\frac{3y^2x^2 - y^4}{(x^2+y^2)^3} \quad \forall\, (x,y) \in \mathbb{R}^2 - \{(0,0)\}$. Quindi ω è chiusa in $\mathbb{R}^2 - \{(0,0)\}$

(b) Sia $\gamma(t) = (\cos t, \sin t) \quad \dot{\gamma}\,(t) = (-\sin t, \cos t)\ t \in [0, 2\pi]$. γ è una curva chiusa in $\mathbb{R}^2 - \{(0,0)\}$ ed è tale che $\int_\gamma \omega = \int_0^{2\pi} -2\sin^4 t - 2\sin^2 \cos^2 t\, dt =$
$= -2\int_0^{2\pi} \sin^2 t dt = -2\pi$. Quindi ω non può essere esatta.

Esercizio 10 Una parametrizzazione del toro è
$\Phi(u,v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$ con $u, v \in [0, 2\pi]$.



$\Phi_u = (-(R + r\cos v)\sin u, (R + r\cos v)\cos u, 0)$
$\Phi_v = (-r\sin v\cos u, -r\sin v\sin u, r\cos v)$
$\Phi_u \wedge \Phi_v = (r(R + r\cos v)\cos u\cos v, r(R + r\cos v)\sin u\cos v, r(R + r\cos v)\sin v)$
$|\Phi_u \wedge \Phi_v| = r(R + r\cos v)$ pertanto si ha che
$Area(T) = \int_0^{2\pi} du \int_0^{2\pi} dv\ r(R + r\cos v) = \int_0^{2\pi} du \int_0^{2\pi} dv\ rR = 4\pi^2 rR$
Calcoliamo ora il volume del toro. Passiamo in coordinate cilindriche cioè poniamo $(x,y,z) = \Psi(\rho,\theta,z) = (\rho\cos\theta, \rho\sin\theta, z)$.
$\Psi^{-1}(T) = \{(\rho,\theta,z) \mid z \in [0,r],\ \theta \in [0,2\pi]\ ,\ R - \sqrt{r^2 - z^2} \le \rho \le R + \sqrt{r - z^2}\}$ quindi
$Vol(T) = \int_T 1 dxdydz = \int_{\Psi^{-1}(T)} \rho\ d\rho d\theta dz = \int_{-r}^{r} dz \int_0^{2\pi} d\theta \int_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} \rho =$
$= 2\pi \int_{-r}^{r} dz \int_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} \rho = \pi \int_{-r}^{r} dz\ \rho^2\Big|_{R-\sqrt{r^2-z^2}}^{R+\sqrt{r^2-z^2}} = 4\pi R \int_{-r}^{r} dz\sqrt{r^2 - z^2} dt =$
$= 8\pi R \int_0^r \sqrt{r^2 - z^2} dz = 8\pi R \int_0^{\frac{\pi}{2}} dt\ r\sqrt{r^2 - r^2\sin^2 t}\cos t\ dt = 8\pi R r^2 \int_0^{\frac{\pi}{2}} \cos^2 t\ dt =$
$= 8\pi R r^2\ \frac{\pi}{4} = 2\pi R r^2$