Università degli Studi Roma Tre - Corso di Laurea in Matematica

AM3 soluzioni tutorato 8

A.A 2008-2009
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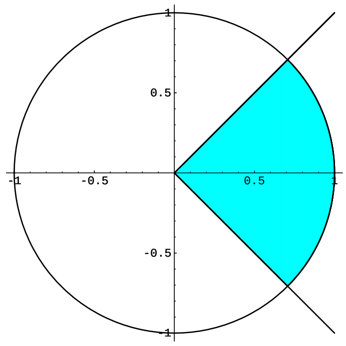
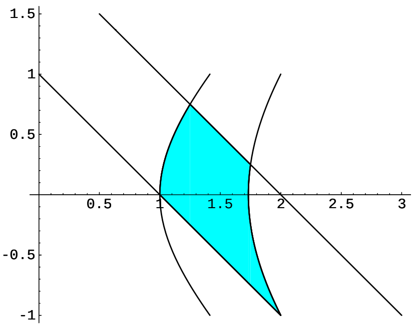
Esercizio 1 $\int_A \frac{x}{1+x^2+y^2}\,dxdy \qquad A = \{(x,y) \in \mathbb{R}^2 \mid |y| \le x \,,\ x^2+y^2 \le 1\}$



Passiamo in coordinate polari cioè poniamo $(x,y) = \Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$.
$\Phi^{-1}(A) = \{(\rho,\theta) \mid 0 \le \rho \le 1 \,,\ -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}\}$ e $|\det J\Phi(\rho,\theta)| = \rho$ quindi per il teorema del cambio di variabile si ha

$$\int_A \frac{x}{1+x^2+y^2}\,dxdy = \int_{\Phi^{-1}(A)} \frac{\rho\cos\theta}{1+\rho^2}\,|\det J\Phi(\rho,\theta)| d\rho d\theta = \int_{\Phi^{-1}(A)} \frac{\rho^2\cos\theta}{1+\rho^2}\,d\rho d\theta =$$
$$= \int_0^1 d\rho \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta\ \frac{\rho^2\cos\theta}{1+\rho^2} = \int_0^1 d\rho \frac{\rho^2}{1+\rho^2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta\ \cos\theta = \int_0^1 d\rho \frac{\rho^2}{1+\rho^2} \sin\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$$
$$= \int_0^1 d\rho \frac{\sqrt{2}\rho^2}{1+\rho^2} = \sqrt{2}\int_0^1 d\rho \left(1 - \frac{1}{1+\rho^2}\right) = \sqrt{2} - \sqrt{2}\arctan\rho\Big|_0^1 = \sqrt{2} - \frac{\sqrt{2}}{4}\pi$$

Esercizio 2 $\int_B (x-y)\log(1+x^2-y^2)\ dxdy \qquad B = \{\ (x,y) \in \mathbb{R}^2 \mid 1-x \le y \le 2-x \,,\ 1 \le x^2-y^2 \le 3\ \}$.
$B = \{(x,y) \mid 1 \le x+y \le 2 \,,\ 1 \le x^2-y^2 \le 3\}$



Poniamo $u = x+y$ e $v = x^2-y^2$ cioè $(x,y) = \Phi(u,v)$ dove
$\Phi(u,v) = \left(\frac{u^2+v}{2u}, \frac{u^2-v}{2u}\right)$.

$$|\det J\Phi| = \begin{vmatrix} \frac{u^2-v}{2u^2} & \frac{1}{2u} \\ \frac{u^2+v}{2u^2} & -\frac{1}{2u} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} - \frac{v}{2u^2} & \frac{1}{2u} \\ \frac{1}{2} + \frac{v}{2u^2} & -\frac{1}{2u} \end{vmatrix} = \left|-\frac{1}{2u}\right| = \frac{1}{2u}.$$

$\Big($si poteva anche osservare che $\det J\Phi^{-1} = \begin{vmatrix} 1 & 1 \\ 2x & -2y \end{vmatrix} = 2x + 2y = 2u$ e quindi

$|\det J\Phi| = \left|\dfrac{1}{\det J\Phi^{-1}}\right| = \dfrac{1}{2u}\Big)$

$\Phi^{-1}(B) = \{(u,v) \in \mathbb{R}^2 \mid 1 \le u \le 2\ ,\ 1 \le v \le 3\}$.

$$\int_B (x-y)\log(1+x^2-y^2)\ dxdy = \int_{\Phi^{-1}(B)} \frac{v}{u}\log(1+v)\,|\det J\phi|\ dudv =$$

$$= \int_1^2 du \int_1^3 dv\ \frac{v}{u}\log(1+v)\frac{1}{2u} = \int_1^3 dv \int_1^2 du \frac{v}{2u^2}\log(1+v) =$$

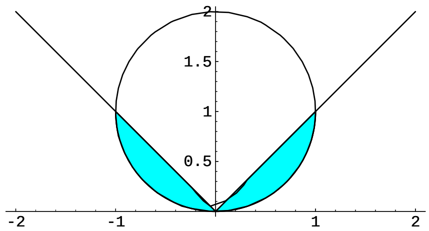
$$= \frac{1}{2}\int_1^3 dv\ -\frac{1}{u}v\log(1+v)\Big|_1^2 = \frac{1}{4}\int_1^3 dv\ v\log(1+v) = \frac{1}{4}\left(\frac{1}{2}v^2\log(1+v)\Big|_1^3 + \right.$$

$$\left. -\frac{1}{2}\int_0^1 \frac{v^2}{1+v}dv\right) = \frac{1}{8}\left(9\log 4 - \log 2 - \int_0^1 v - 1 + \frac{1}{1+v}dv\right) =$$

$$\frac{1}{8}\left(17\log 2 - \frac{1}{2}v^2 + v - \log(1+v)\Big|_1^3\right) = \frac{1}{8}\left(17\log 2 - \frac{9}{2} + \frac{1}{2} + 2 - \log 4 + \log 2\right)$$

$$= \frac{1}{8}(16\log 2 - 2) = 2\log 2 - \frac{1}{4}$$

Esercizio 3 $f(x,y) = |x|^3 - y \quad C = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 2y\ ,\ 0 \le y \le |x|\}$



Siccome f è pari nella variabile x allora $\int_C f(x,y)dxdy = 2\int_{C'} f(x,y)\ dxdy$ dove $C' = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 2y\ ,\ 0 \le y \le x\ ,\ x \ge 0\}$

Passando in coordinate polari $(x,y) = \Phi(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta)$ si ha $\Phi^{-1}(C') = \{(\rho,\theta) \mid 0 \le \theta \le \frac{\pi}{4},\ 0 \le \rho \le 2\sin\theta\}$ e $|\det J\Phi(\rho,\theta)| = \rho$ quindi

$$\int_{C'} f(x,y)dxdy = \int_{\Phi^{-1}(C')} f(\rho\cos\theta, \rho\sin\theta)\rho\ d\rho d\theta = \int_0^{\frac{\pi}{4}} d\theta \int_0^{2\sin\theta} d\rho\ \rho^4\cos^3\theta - \rho^2\sin\theta$$

$$= \int_0^{\frac{\pi}{4}} d\theta \frac{1}{5}\rho^5\cos^3\theta - \frac{1}{3}\rho^3\sin\theta\Big|_0^{2\sin\theta} = \int_0^{\frac{\pi}{4}} d\theta\ \frac{32}{5}\sin^5\theta\cos^3\theta - \frac{8}{3}\sin^4\theta \overset{(y=\sin\theta)}{=}$$

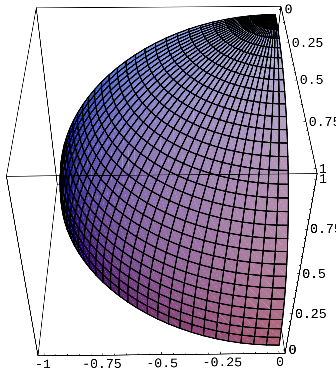
$$= \frac{32}{5}\int_0^{\frac{1}{\sqrt{2}}} t^5(1-t^2)dt - \frac{8}{3}\int_0^{\frac{\pi}{4}} \sin^4\theta\ d\theta = \frac{32}{5}\left(\frac{1}{6}t^6 - \frac{1}{8}t^8\right)\Big|_0^{\frac{1}{\sqrt{2}}} - \frac{8}{3}\int_0^{\frac{\pi}{4}} \sin^4\theta\ d\theta$$

$$= \frac{2}{15} - \frac{1}{20} - \frac{8}{3}\left(-\cos\theta\sin^3\theta\Big|_0^{\frac{\pi}{4}} + 3\int_0^{\frac{\pi}{4}} \cos^2\theta\sin^2\theta\ d\theta\right) = \frac{1}{12} + \frac{2}{3} - 2\int_0^{\frac{\pi}{4}} \sin^2 2\theta\ d\theta =$$

$$= \frac{3}{4} - \int_0^{\frac{\pi}{2}} \sin^2\theta\ d\theta = \frac{3}{4} - \frac{\pi}{4}$$

Quindi $\int_C f(x,y)dxdy = 2\int_{C'} f(x,y)\ dxdy = \frac{3}{2} - \frac{\pi}{2}$

Esercizio 4 $\int_D xy^2z\ dxdydz \quad D = \{\ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\ ,\ x \ge 0\ ,\ y \le 0\ ,\ z \ge 0\}$.



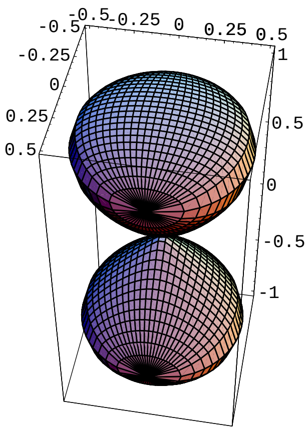
Passiamo in coordinate sferiche cioè poniamo
$(x, y, z) = \Phi(\rho, \theta, \phi) = (\rho \cos\theta \sin\phi, \rho \sin\theta \sin\phi, \rho \cos\phi)$.
$\Phi^{-1}(D) = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le \rho \le 1 \ , \ -\frac{\pi}{2} \le \theta \le 0 \ , \ 0 \le \phi \le \frac{\pi}{2}\}$ e
$|\det J\Phi(\rho, \theta, \phi)| = \rho^2 \sin\phi$ quindi per il teorema del cambio di variabile si ha

$$\int_D xy^2 z \; dxdydz = \int_{\Phi^{-1}(D)} \rho^6 \cos\theta \sin^2\theta \sin^4\phi \cos\phi \; d\rho \; d\theta \; d\phi =$$

$$= \int_0^1 d\rho \int_{-\frac{\pi}{2}}^0 d\theta \int_0^{\frac{\pi}{2}} d\phi \; \rho^6 \cos\theta \sin^2\theta \sin^4\phi \cos\phi = \frac{1}{5} \int_0^1 d\rho \int_{-\frac{\pi}{2}}^0 d\theta \rho^6 \cos\theta \sin^2\theta \sin^5\phi \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{5} \int_0^1 d\rho \int_{-\frac{\pi}{2}}^0 d\theta \; \rho^6 \cos\theta \sin^2\theta = \frac{1}{15} \int_0^1 d\rho \; \rho^6 \sin^3\theta \Big|_{-\frac{\pi}{2}}^0 = \frac{1}{15} \int_0^1 d\rho \; \rho^6 = \frac{1}{105}$$

Esercizio 5 $\displaystyle\int_E 3x^2 + |y| - 2z^2 \; dxdydz \quad E = \{ \ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \le z^2(1 - z^2)\}.$

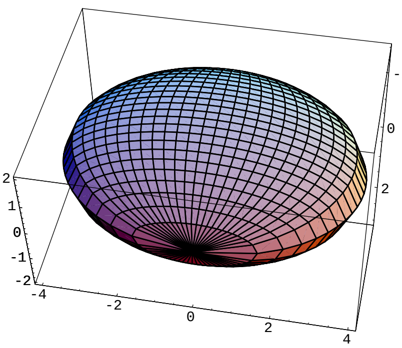


Passiamo in coordinate cilindriche ponendo $(x, y, z) = \Phi(\rho, \theta, z) = (\rho \cos\theta, \rho \sin\theta, z)$.
$\Phi^{-1}(\theta) = \{(\rho, \theta, z) \mid z \in [-1, 1] \ , \ \theta \in [0, 2\pi], 0 \le \rho \le |z|\sqrt{1 - z^2}\}$ e $|\det J\Phi(\rho, \theta, z)| =$
ρ quindi per il teorema del cambio di variabile si ha

$$\int_E 3x^2 + |y| - 2z^2 \; dxdydz = \int_{\Phi^{-1}(E)} 3\rho^3 \cos^2\theta + \rho^2 |\sin\theta| - 2\rho z^2 d\rho \; d\theta \; dz =$$

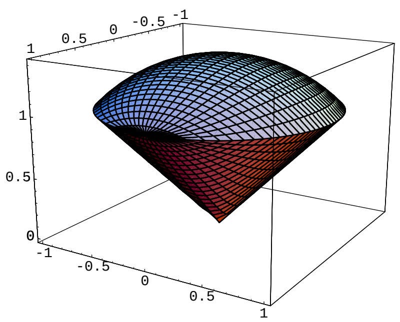
$$= \int_{-1}^{1} dz \int_0^{2\pi} d\theta \int_0^{|z|\sqrt{1-z^2}} 3\rho^3 \cos^2\theta + \rho^2 |\sin\theta| - 2\rho z^2 =$$
$$= 2\int_0^1 dz \int_0^{2\pi} d\theta \int_0^{|z|\sqrt{1-z^2}} 3\rho^3 \cos^2\theta + \rho^2 |\sin\theta| - 2\rho z^2 =$$
$$= 2\int_0^1 dz \int_0^{2\pi} d\theta \frac{3}{4}\rho^4 \cos^2\theta + \frac{1}{3}\rho^3 |\sin\theta| - \rho^2 z^2 \Big|_0^{|z|\sqrt{1-z^2}} =$$
$$= 2\int_0^1 dz \int_0^{2\pi} d\theta \frac{3}{4} z^4 (1-z^2)^2 \cos^2\theta + \frac{1}{3} z^3 (1-z^2)^{\frac{3}{2}} |\sin\theta| - z^4(1-z^2) =$$
$$= 2\int_0^1 dz \frac{3}{4}\pi z^4 (1 - 2z^2 + z^4) + \frac{1}{3} z^3 (1-z^2)^{\frac{3}{2}} \int_0^{2\pi} |\sin\theta| d\theta - 2\pi z^4 (1-z^2) =$$
$$= 2\int_0^1 -\frac{5}{4}\pi z^4 + \frac{1}{2}\pi z^6 + 3\frac{\pi}{4} z^8 + \frac{4}{3} z^3 (1-z^2)^{\frac{3}{2}} dz = \pi\left(-\frac{1}{2} + \frac{1}{7} + \frac{1}{6}\right) +$$
$$+ \frac{8}{3}\int_0^1 z^3 (1-z^2)^{\frac{3}{2}}\, dz = -\frac{4}{21}\pi + \frac{8}{3}\int_0^{\frac{\pi}{2}} \sin^3 t\ \cos^4 t\ dt = -\frac{4}{21}\pi +$$
$$= \frac{8}{3}\int_0^1 du\ u^4 (1-u^2) = -\frac{4}{21}\pi + \frac{8}{3}\int_0^1 du\ u^4 - u^6 = -\frac{4}{21}\pi + \frac{8}{3}\left(\frac{1}{5} - \frac{1}{7}\right) =$$
$$= -\frac{4}{21}\pi + \frac{16}{105}$$

Esercizio 6 Sia F l' ellissoide in $\mathbb{R}^3$ di equazione $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$
Poniamo $(x, y, z) = \Phi(u, v, w) = (a\ u;,\ b\ v\ ,\ c\ w)$



$\Phi^{-1}(F) = (u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1$ e $|detJ\Phi| = abc$ quindi
$Vol(F) = \int_F 1dxdydz = \int_{\Phi^{-1}(F)} abc\ dudvdw = abc\ Vol(\Phi^{-1}(F)) = \frac{4}{3}\pi abc$

Esercizio 7 $f(x, y, z) = z\sqrt{z^2 - x^2 - y^2}$ $\quad G = \{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2} \leq z \leq \sqrt{2 - x^2 - y^2}$



In cordinate sferiche $\Phi^{-1}(G)=\{(\rho,\theta,\phi)\mid 0\le\rho\le\sqrt{2}\ \theta\in[0,2\pi]\ 0\le\phi\le\frac{\pi}{4}\}$ quindi

$$\int_G f(x,y,z)=\int_{\Phi^{-1}(G)}\sqrt{\rho^2\cos^2\phi-\rho^2\sin^2\phi}\ \rho^3\sin\phi\cos\phi\ d\rho\ d\theta\ d\phi=$$

$$=\frac{1}{2}\int_{\Phi^{-1}(G)}\sqrt{\cos 2\phi}\ \rho^4\sin 2\phi\ d\rho\ d\theta\ d\phi=\frac{1}{2}\int_0^{\sqrt{2}}d\rho\ \rho^4\int_0^{\frac{\pi}{4}}d\phi\int_0^{2\pi}d\theta\ \sqrt{\cos 2\phi}\ \sin 2\phi=$$

$$=\pi\int_0^{\sqrt{2}}d\rho\ \rho^4\int_0^{\frac{\pi}{4}}d\phi\ \sqrt{\cos 2\phi}\ \sin 2\phi\ =\frac{\pi}{2}\int_0^{\sqrt{2}}d\rho\ \rho^4\int_0^{\frac{\pi}{2}}dt\ \sqrt{\cos t}\ \sin t=$$

$$=\frac{\pi}{2}\int_0^{\sqrt{2}}d\rho\ \rho^4\left(-\frac{2}{3}\cos^{\frac{3}{2}}t\right)\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{3}\int_0^{\sqrt{2}}\rho^4\ d\rho=\frac{\pi}{3}\ \frac{1}{5}\rho^5\Big|_0^{\sqrt{2}}=\frac{4\sqrt{2}}{15}\pi$$

Esercizio 8 $\gamma(t)=(\cos t,t-\sin t)\quad t\in[0,2\pi]$
$\dot{\gamma}\ (t)=(-\sin t,1-\cos t)\quad t\in[0,2\pi]$
$|\ \dot{\gamma}\ (t)\ |=\sqrt{\sin^2 t+(1-\cos t)^2}=\sqrt{2-2\cos t}\neq 0\ \forall t\in(0,2\pi)$ quindi γ è una curva regolare

$$l(\gamma)=\int_0^{2\pi}|\ \dot{\gamma}\ (t)\ |=\int_0^{2\pi}\sqrt{2-2\cos t}=\sqrt{2}\int_0^{2\pi}\sqrt{1-\cos t}\ dt=\sqrt{2}\int_0^{2\pi}\sqrt{2\sin^2\frac{t}{2}}=$$

$$=2\int_0^{2\pi}\sin\frac{t}{2}=-4\cos\frac{t}{2}\Big|_0^{2\pi}=8$$

Esercizio 9 $\alpha(t)=(t,t^2,t^3)\quad t\in[0,\frac{\pi}{6}]\qquad \dot{\alpha}\ (t)=(1,2t,3t^2)$
$|\ \dot{\alpha}\ (t)\ |=\sqrt{1+4t^2+9t^4}$

$$\int_\alpha f(x,y,z)\ ds=\int_0^{\frac{\pi}{6}}f(\alpha(t))|\ \dot{\alpha}\ (t)\ |=\int_0^{\frac{\pi}{6}}\frac{1}{\cos^3 t\ \sqrt{1+4t^2+9t^4}}\sqrt{1+4t^2+9t^4}dt=$$

$$=\int_0^{\frac{\pi}{6}}\frac{1}{\cos^3 x}dx$$

Ma $\displaystyle\int_0^{\frac{\pi}{6}}\frac{1}{\cos^3 x}dx=\tan x\frac{1}{\cos x}\Big|_0^{\frac{\pi}{6}}-\int_0^{\frac{\pi}{6}}\frac{\sin x}{\cos^2 x}\tan x dx=\frac{2}{3}-\int_0^{\frac{\pi}{6}}\frac{\sin^2 x}{\cos^3 x}dx=$

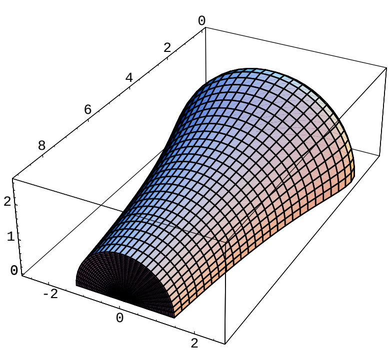
$$=\frac{2}{3}+\int_0^{\frac{\pi}{6}}\frac{1}{\cos x}dx-\int_0^{\frac{\pi}{6}}\frac{1}{\cos^3 x}dx\Longrightarrow\int_0^{\frac{\pi}{6}}\frac{1}{\cos^3 x}dx=\frac{1}{3}+\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{1}{\cos x}dx\overset{t=\sin x}{=}$$

$$=\frac{1}{3}+\int_0^{\frac{1}{2}}\frac{1}{1-t^2}dt=\frac{1}{3}+\frac{1}{4}\int_0^{\frac{1}{2}}dt\frac{1}{1+t}+\frac{1}{1-t}=\frac{1}{3}+\frac{1}{4}\log\left(\frac{1+t}{1-t}\right)\Big|_0^{\frac{1}{2}}=\frac{1}{3}+\frac{1}{4}\log 3$$

Quindi $l(\gamma)=\frac{1}{3}+\frac{1}{4}\log 3$

Esercizio 10 Sia $H=\{(x,y,z)\in\mathbb{R}^3\mid y^2+z^2\le\frac{16\arctan^2 x}{\sqrt[3]{1+x^2}}\ ,\ z>0\ ,\ x\in[0,r]\}$.

Calcoliamo $\displaystyle\int_H z^2y^2-y^4+\frac{\sin(\arctan x)}{(1+x^2)^{\frac{2}{3}}}\ dxdydz$



Passiamo in coordinate cilindriche ponendo $(x, y, z) = \Phi(x, \rho, \theta) = (x, \rho\cos\theta, \rho\sin\theta)$

$\Phi^{-1}(H) = \{(x, \rho, \theta) \mid x \in [0, r]\ ,\ \theta \in [0, \pi]\ , \rho \leq 0 \leq \frac{4\arctan x}{\sqrt[6]{1+x^2}}\}$ e $|\det J\Phi| = \rho$ quindi

$$\int_H z^2y^2 - y^4 + \frac{\sin(\arctan x)}{(1+x^2)^{\frac{2}{3}}}\ dxdydz =$$

$$= \int_0^r dx \int_0^\pi d\theta \int_0^{\frac{4\arctan x}{\sqrt[6]{1+x^2}}} \rho^5 \sin^2\theta\cos^2\theta - \rho^5\cos^4\theta + \rho\frac{\sin(\arctan x)}{(1+x^2)^{\frac{2}{3}}}\ d\rho =$$

$$= \int_0^r dx \int_0^\pi d\theta \frac{1}{6}\rho^6\sin^2\theta\cos^2\theta - \frac{1}{6}\rho^6\cos^4\theta + \frac{1}{2}\rho^2\frac{\sin(\arctan x)}{(1+x^2)^{\frac{2}{3}}}\Big|_0^{\frac{4\arctan x}{\sqrt[6]{1+x^2}}} =$$

$$= \int_0^r dx \int_0^\pi \frac{2^{11}}{3}\frac{\arctan^6 x}{1+x^2}\sin^2\theta\cos^2\theta - \frac{2^{11}}{3}\frac{\arctan^6 x}{1+x^2}\cos^4\theta + 8\frac{\arctan^2 x}{\sqrt[3]{1+x^2}}\frac{\sin(\arctan x)}{(1+x^2)^{\frac{2}{3}}} =$$

$$= \frac{2^{11}}{3}\int_0^r dx\frac{\arctan^6 x}{1+x^2}\int_0^\pi d\theta\ \sin^2\theta\cos^2\theta - \cos^4\theta + 8\pi\int_0^r dx\frac{\arctan^2 x}{1+x^2}\sin(\arctan x) =$$

$$= \frac{2^{11}}{3}\int_0^r dx\frac{\arctan^6 x}{1+x^2}\int_0^\pi d\theta\ \cos^2\theta - 2\cos^4\theta + 8\pi\int_0^{\arctan r} t^2\sin t\ dt =$$

$$= \frac{2^{11}}{3}\int_0^r dx\frac{\arctan^6 x}{1+x^2}\left(\frac{\pi}{2} - 2\sin\theta\cos^3\theta\Big|_0^\pi - 6\int_0^\pi \sin^2\theta\cos^2\theta d\theta\right) + 8\pi\int_0^{\arctan r} t^2\sin t\ dt =$$

$$= \frac{2^{11}}{3}\int_0^r dx\frac{\arctan^6 x}{1+x^2}\left(\frac{\pi}{2} - \frac{3}{2}\int_0^\pi \sin^2 2\theta d\theta\right) + 8\pi\left(-t^2\cos t\Big|_0^{\arctan r} + \int_0^{\arctan r} 2t\cos t\ dt\right)$$

$$= \frac{2^{11}}{3}\int_0^r dx\frac{\arctan^6 x}{1+x^2}\left(\frac{\pi}{2} - \frac{3}{4}\pi\right) + 8\pi\left(-\arctan^2 r\cos(\arctan r) + 2t\sin t\ \Big|_0^{\arctan r} - 2\int_0^{\arctan r}\sin t dt\right)$$

$$= -\frac{2^9}{3}\pi\int_0^r dx\frac{\arctan^6 x}{1+x^2} + 8\pi\left(-\arctan^2 r\cos(\arctan r) + 2\arctan r\sin(\arctan r) - 2\int_0^{\arctan r}\sin t dt\right)$$

$$= -\frac{2^9}{21}\pi\arctan^7 r + 8\pi\left(-\arctan^2 r\cos(\arctan r) + 2\arctan r\sin(\arctan r) + 2\cos(\arctan r) - 2\right) =$$

$$= -\frac{2^9}{21}\pi\arctan^7 r - 8\pi\arctan^2 r\cos\arctan r + 16\pi\arctan r\sin\arctan r + 16\pi\cos\arctan r - 16\pi$$

In particolare $\lim_{r\to\infty}\int_H z^2y^2 - y^4 + \frac{\sin(\arctan x)}{(1+x^2)^{\frac{2}{3}}}\ dxdydz = -\frac{4}{21}\pi^8 + 8\pi^2 - 16\pi$