Università degli Studi Roma Tre - Corso di Laurea in Matematica

AM3 soluzioni tutorato 7

A.A 2008-2009
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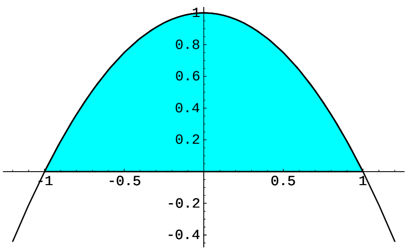
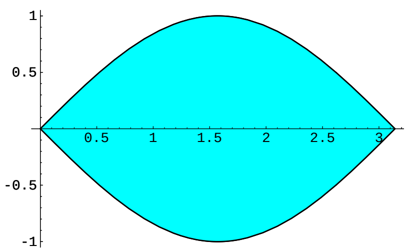
Esercizio 1 Calcolare $\int_A x^4 + 2x^2y\, dxdy \qquad A = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 - x^2\}$
$A = \{(x,y) \in \mathbb{R}^2 \mid -1 \le x \le 1\ ,\ 0 \le y \le 1 - x^2\}$



$$\int_A x^4 + 2x^2y\, dxdy = \int_{-1}^{1} dx \int_0^{1-x^2} dy\; x^4 + 2x^2y = \int_{-1}^{1} dx \left(x^4y + x^2y^2\right)\Big|_0^{1-x^2} =$$
$$= \int_{-1}^{1} x^4 - x^6 + x^2(1-x^2)^2 dx = \int_{-1}^{1} x^4 - x^6 + x^2 - 2x^4 + x^6 dx =$$
$$= \int_{-1}^{1} x^2 - x^4 dx = 2\int_0^1 x^2 - x^4 dx = \frac{2}{3}x^3 - \frac{2}{5}x^5\Big|_0^1 = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$$

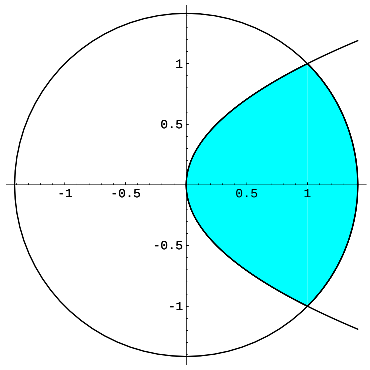
Esercizio 2 $f(x,y) = 2y^2\cos^4 x - y \qquad B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi,\ |y| \le \sin x\}$

$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \pi,\ -\sin x \le y \le \sin x\}$



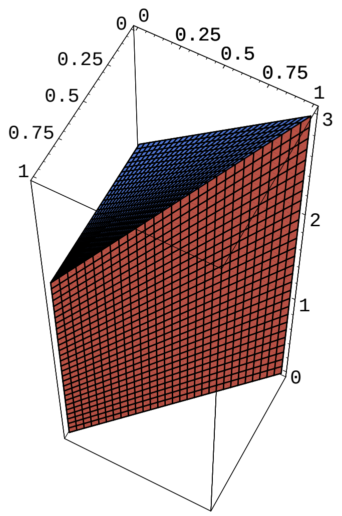
$$\int_B 2y^2\cos^4 x - y\, dxdy = \int_0^{\pi} dx \int_{-\sin x}^{\sin x} dy\; 2y^2\cos^4 x - y =$$
$$= \int_0^{\pi} dx\; \frac{2}{3}y^3\cos^4 x - \frac{1}{2}y^2\Big|_{-sinx}^{\sin x} = \frac{4}{3}\int_0^{\pi} \sin^3 x\cos^4 x\, dx =$$
$$= \frac{4}{3}\int_0^{\pi} \sin x(1-\cos^2 x)\cos^4 x\, dx \overset{(t=\cos x)}{=} \frac{4}{3}\int_{-1}^{1}(1-t^2)t^4 dt =$$
$$= \frac{8}{3}\int_0^1 t^4 - t^6 dt = \frac{8}{3}\left(\frac{1}{5}t^5 - \frac{1}{7}t^7\right)\Big|_0^1 = \frac{8}{3}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{8}{3}\,\frac{2}{35} = \frac{16}{105}$$

Esercizio 3 $C = \{ \ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 2, \ x \geq y^2 \ \}$
$C = \{ \ (x,y) \in \mathbb{R}^2 \mid \ -1 \leq y \leq 1 \ , \ y^2 \leq x \leq \sqrt{2-y^2} \ \}.$



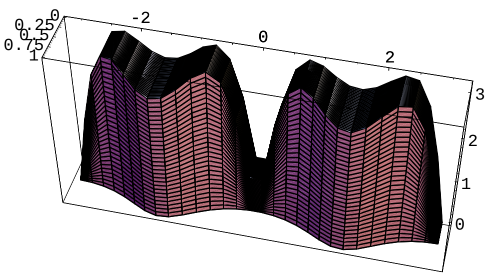
$$Area(C) = \int_C 1 \ dxdy = \int_{-1}^{1} dy \int_{y^2}^{\sqrt{2-y^2}} dx \ 1 = \int_{-1}^{1} dy \ \sqrt{2-y^2} - y^2 =$$
$$= \int_{-1}^{1} dy \ \sqrt{2-y^2} - \int_{-1}^{1} y^2 dy \overset{(\ y=\sqrt{2}\sin t \ )}{=} 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 t \ dt - \frac{1}{3}y^3 \Big|_{-1}^{1} =$$
$$= 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2t + 1}{2} dt - \frac{2}{3} = \frac{1}{2}\sin(2t) + t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} - \frac{2}{3} = 1 + \frac{\pi}{2} - \frac{2}{3} = \frac{1}{3} + \frac{\pi}{2}$$
$$\int_C y e^{y^2} + x \ \ dxdy = \int_{-1}^{1} dy \int_{y^2}^{\sqrt{2-y^2}} dx \ (y e^{y^2} + x) = \int_{-1}^{1} dy \ xye^{y^2} + \frac{1}{2}x^2 \Big|_{y^2}^{\sqrt{2-y^2}} =$$
$$\int_{-1}^{1} y\sqrt{2-y^2}e^{y^2} - y^3 e^{y^2} + \frac{1}{2}(2-y^2) - \frac{1}{2}y^4 \ dy = 2\int_0^1 \frac{1}{2}(2-y^2) - \frac{1}{2}y^4 dy =$$
$$= \int_0^1 (2-y^2) - y^4 dy = 2 - \frac{1}{3}y^3 - \frac{1}{5}y^5 \Big|_0^1 = 2 - \frac{1}{3} - \frac{1}{5} = \frac{22}{15}$$

Esercizio 4 $D = \{ \ (x,y,z) \in \mathbb{R}^3 \mid x + y \leq 1, \ 0 \leq z \leq 2 + y, \ x \geq 0, \ y \geq 0 \ \}.$
$D = \{ \ (x,y,z) \in \mathbb{R}^3 \mid 0 \leq x \leq 1 \ , \ 0 \leq y \leq 1 - x \ , \ 0 \leq z \leq 2 + y \ \}$



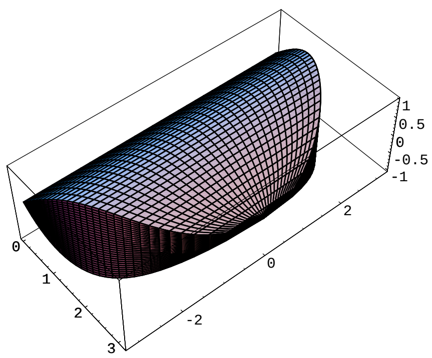
$$\int_D xy+z\,dxdydz = \int_0^1 dx \int_0^{1-x} dy \int_0^{2+y} dz\,(xy+z) = \int_0^1 dx\int_0^{1-x} xyz + \frac{1}{2} z^2\Big|_0^{2+y} =$$
$$\int_0^1 dx\int_0^{1-x} dy\; xy(2+y)+\frac{1}{2}(2+y)^2 = \int_0^1 dx \int_0^{1-x} dy\; 2xy+xy^2+\frac{1}{2}(2+y)^2 dy =$$
$$= \int_0^1 xy^2+\frac{1}{3} xy^3+\frac{1}{6}(2+y)^3\Big|_0^{1-x} = \int_0^1 dx\; x(1-x)^2+\frac{1}{3} x(1-x)^3+\frac{1}{6}(3-x)^3-\frac{4}{3} =$$
$$=\int_0^1 x-2x^2+x^3+\frac{1}{3} x - x^2+x^3-\frac{1}{3}x^4+\frac{1}{6}(3-x)^3-\frac{4}{3}\,dx=$$
$$=\int_0^1 \frac{4}{3}x-3x^2+2x^3-\frac{1}{3}x^4+\frac{1}{6}(3-x)^3-\frac{4}{3}dx = \frac{2}{3}x^2-x^3+\frac{1}{2}x^4-\frac{1}{15}x^5+$$
$$-\frac{1}{24}(3-x)^4-\frac{4}{3}\Big|_0^1 = \frac{2}{3}-1+\frac{1}{2}-\frac{1}{15}-\frac{2}{3}+\frac{27}{8}-\frac{4}{3} = -\frac{1}{2}-\frac{1}{15}+\frac{27}{8}-\frac{4}{3}=$$
$$=\frac{-60-8+405-160}{120}=\frac{177}{120}=\frac{59}{40}$$

Esercizio 5 $E = \{\ (x,y,z)\in\mathbb{R}^3 \mid |z| \le |\sin y| e^{x(1+cos^2 y)},\ 0\le x\le \frac{1}{1+\cos^2 y}, |y|\le\pi\}$



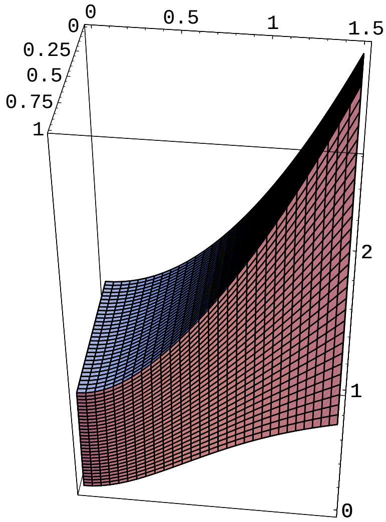
$$Vol(E) = \int_E 1dxdydz = \int_{-\pi}^{\pi} dy \int_0^{\frac{1}{1+\cos^2 y}} dx \int_{-|\sin y| e^{x(1+cos^2 y)}}^{|\sin y| e^{x(1+cos^2 y)}} dz\ 1 =$$
$$= 2\int_{-\pi}^{\pi} dy \int_0^{\frac{1}{1+\cos^2 y}} dx |\sin y|\ e^{x(1+cos^2 y)} = 2\int_{-\pi}^{\pi} dy\ \frac{|\sin y|}{1+\cos^2 y} e^{x(1+cos^2 y)}\Big|_0^{\frac{1}{1+\cos^2 y}} =$$
$$= 2(e-1)\int_{-\pi}^{\pi} \frac{|\sin y|}{1+\cos^2 y} dy = 4(e-1)\int_0^{\pi} \frac{\sin y}{1+\cos^2 y} dy = 4(e-1)\int_{-1}^{1} \frac{1}{1+t^2} dt =$$
$$= 4(e-1)\arctan t\Big|_{-1}^{1} = 2\pi(e-1)$$

Esercizio 6 $F = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2\le\pi^2,\ x\ge 0,\ |z|\le\sin x\}$
$F = \{(x,y,z)\in\mathbb{R}^3 \mid 0\le x\le\pi,\ -\sqrt{\pi^2-x^2}\le y\le\sqrt{\pi^2-x^2}\ ,\ -\sin x\le z\le\sin x\}$



$$\int_F (1+z)\sqrt[4]{\pi^2y^2-x^2y^2}dxdydz = \int_{\{x^2+y^2\le\pi^2,\ x\ge0\}} dxdy \int_{-\sin x}^{\sin x} dz\ (1+z)\sqrt[4]{\pi^2y^2-x^2y^2} =$$
$$= 2\int_0^\pi dx \int_{-\sqrt{\pi^2-x^2}}^{\sqrt{\pi^2-x^2}} dy \int_0^{\sin x} dz\sqrt{|y|}\sqrt[4]{\pi^2-x^2} = 4\int_0^\pi dx \int_0^{\sqrt{\pi^2-x^2}} dy\sqrt{y}\ \sqrt[4]{\pi^2-x^2}\ \sin x$$
$$= \frac{8}{3}\int_0^\pi (\pi^2-x^2)\sin x = \frac{8}{3}\int_0^\pi dx\ \pi^2\sin x - x^2\sin x = \frac{16}{3}\pi^2 - \frac{8}{3}\int_0^\pi x^2\sin x dx =$$
$$= \frac{16}{3}\pi^2 + \frac{8}{3}\left(x^2\cos x\Big|_0^\pi - 2\int_0^\pi x\cos x\right) = \frac{16}{3}\pi^2 - \frac{8}{3}\pi^2 - \frac{16}{3}\left(x\sin x\Big|_0^\pi - \int_0^\pi \sin x\right) =$$
$$= \frac{8}{3}\pi^2 + \frac{16}{3}\int_0^\pi \sin x\ dx = \frac{8}{3}\pi^2 - \frac{16}{3}\cos x\Big|_0^\pi = \frac{8}{3}\pi^2 + \frac{32}{3}$$

Esercizio 7 $G = \{(x,y,z)\in\mathbb{R}^3 \mid 0\le y\le r,\ 0\le x\le \frac{1}{1+y^2},\ |z|\le \frac{\pi}{2}(1+y^2)\}.$

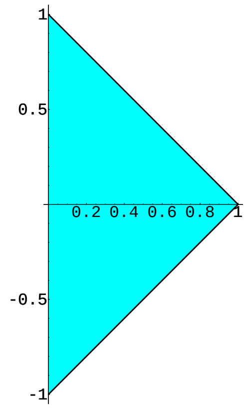


$$\int_G f(x,y,z)dxdydz = 2\int_0^r dy \int_0^{\frac{1}{1+y^2}} dx \int_0^{\frac{\pi}{2}(1+y^2)} dz\ \frac{\sqrt{y(r-y)}}{1+(y^2x+x)^2}\cos\left(\frac{z}{1+y^2}\right) =$$
$$= 2\int_0^r dy \int_0^{\frac{1}{1+y^2}} dx\frac{(1+y^2)\sqrt{y(r-y)}}{1+(y^2x+x)^2}\sin\left(\frac{z}{1+y^2}\right)\Big|_0^{\frac{\pi}{2}(1+y^2)} =$$
$$= 2\int_0^r dy \int_0^{\frac{1}{1+y^2}} dx\frac{(1+y^2)\sqrt{y(r-y)}}{1+(y^2x+x)^2} = 2\int_0^r dy \int_0^{\frac{1}{1+y^2}} dx\frac{(1+y^2)\sqrt{y(r-y)}}{1+x^2(y^2+1)^2} =$$
$$= 2\int_0^r dx\sqrt{y(r-y)}\arctan(x(1+y^2))\Big|_0^{\frac{1}{1+y^2}} = \frac{\pi}{2}\int_0^r \sqrt{y(r-y)}dy \overset{(y=r\sin^2 t)}{=}$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sqrt{r\sin^2 t(r-r\sin^2 t)}2r\sin t\cos t = \pi r^2\int_0^{\frac{\pi}{2}} \sin^2 t\ \cos^2 t\ dt =$$
$$= \frac{\pi r^2}{4}\int_0^{\frac{\pi}{2}} dt\sin^2(2t)dt = \frac{\pi r^2}{8}\int_0^\pi \sin^2 t\ dt = \frac{\pi^2r^2}{16}$$

In particolare si ha $\int_I f(x,y,z)dxdydz = 1 \Longleftrightarrow \frac{\pi^2r^2}{16} = 1 \Longleftrightarrow r = \frac{4}{\pi}$

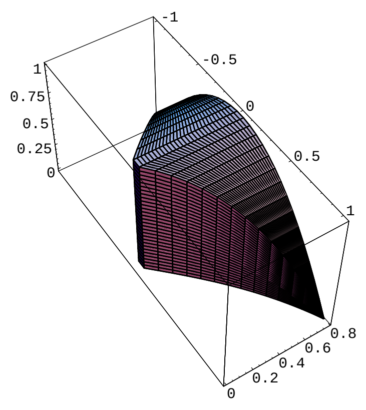
Esercizio 8 $\int_H xy\log(1+x+y)dxdy \qquad H = \{(x,y)\in\mathbb{R}^2 \mid x\ge0,\ |y|\le 1-x\}$

$H = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1,\ x-1\le y\le 1-x\}$



$$\int_H xy\log(1+x+y)dxdy = \int_0^1 dx \int_{x-1}^{1-x} dx\ xy\log(1+x+y) =$$
$$\int_0^1 dx\ x\int_{x-1}^{1-x} dy\ y\log(1+x+y) = \int_0^1 dx\ x\left(\frac{1}{2}y^2\log(1+x+y)\Big|_{x-1}^{1-x} - \int_{x-1}^{1-x}\frac{1}{2}\frac{y^2}{1+x+y}\right)$$
$$= \frac{1}{2}\int_0^1 dx\ x(1-x)^2\log 2 - x(x-1)^2\log(2x) - \frac{1}{2}\int_0^1 dx\ x\int_{x-1}^{1-x} dy\frac{y^2-(x+1)^2+(x+1)^2}{1+x+y}$$
$$= \int_0^1 dx\ -\frac{1}{2}x(x-1)^2\log x - \frac{1}{2}\int_0^1 dx\ x\int_{x-1}^{1-x} dy\ y-x-1+\frac{(x+1)^2}{1+x+y}dy =$$
$$= \int_0^1 dx\ -\frac{1}{2}x(x-1)^2\log x + x(1-x^2) - \frac{1}{2}x(x+1)^2\log(1+x+y)\Big|_{x-1}^{1-x} =$$
$$= \int_0^1 dx\ -\frac{1}{2}x(x-1)^2\log x + x(1-x^2) - \frac{1}{2}x(x+1)^2\log 2 + \frac{1}{2}x(x+1)^2\log 2x =$$
$$= \int_0^1 dx - \frac{1}{2}x(x-1)^2\log x + x(1-x^2) + \frac{1}{2}x(x+1)^2\log x\ dx =$$
$$= \int_0^1 dx\ x - x^3 + 2x^2\log x = \frac{1}{2} - \frac{1}{4} + \int_0^1 2x^2\log x = \frac{1}{4} + \frac{2}{3}x^2\log x\Big|_0^1 - \frac{2}{3}\int_0^1 x^2dx =$$
$$= \frac{1}{4} - \frac{2}{3}\ \frac{1}{3} = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$$

Esercizio 9 $I = \{(x,y,z)\in\mathbb{R}^3 \mid |\arctan x| \le y \le \frac{\pi}{4},\ 0\le z\le 1-x^2\}$.



$$I = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le y \le \frac{\pi}{4},\ -\tan y \le x \le \tan y,\ 0 \le z \le 1-x^2\}$$

$$\int_I \frac{e^{\cos^2 y}}{(1+x^2)^2}\, dxdydz = \int_0^{\frac{\pi}{4}} dy \int_{-\tan y}^{\tan y} dx \int_0^{1-x^2} dz\ \frac{e^{\cos^2 y}}{(1+x^2)^2} =$$

$$= \int_0^{\frac{\pi}{4}} dy \int_{-\tan y}^{\tan y} dx \frac{1-x^2}{(1+x^2)^2} e^{\cos^2 y} = 2\int_0^{\frac{\pi}{4}} dy\ e^{\cos^2 y} \int_0^{\tan y} dx \frac{1-x^2}{(1+x^2)^2} =$$

$$= 2\int_0^{\frac{\pi}{4}} dy\ e^{\cos^2 y} \int_0^{\tan y} dx \frac{1}{1+x^2} - \frac{2x^2}{(1+x^2)^2} = 2\int_0^{\frac{\pi}{4}} dy e^{\cos^2 y} \left(y - \int_0^{\tan y} x \frac{2x}{(1+x^2)^2} \right)$$

$$= 2\int_0^{\frac{\pi}{4}} dy e^{\cos^2 y} \left(y + \frac{x}{1+x^2}\Big|_0^{\tan y} - \int_0^{\tan y} \frac{1}{1+x^2} dx \right) = 2\int_0^{\frac{\pi}{4}} \frac{\tan y}{1+\tan^2 y} e^{\cos^2 y} dy =$$

$$= \int_0^{\frac{\pi}{4}} \sin 2t\ e^{\cos^2 y} dy = \int_0^{\frac{\pi}{4}} 2\sin y \cos y\ e^{\cos^2 y} dt = 2\int_{\frac{1}{\sqrt{2}}}^{1} te^{t^2} dt = e^{t^2}\Big|_{\frac{1}{\sqrt{2}}}^{1} = e - \sqrt{e}$$