Università degli Studi Roma Tre - Corso di Laurea in Matematica

AM3 soluzioni tutorato 10

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Esercizio 2 Sia $\omega = zdx + (x+z)dy + (y+e^x)dz$ e sia $\gamma : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R}^3$ $\gamma(t) = (\cos t, t, \sin^2 t)$.
Calcoliamo $\displaystyle\int_\gamma \omega$.

$\gamma(t) = (\cos t, t, \sin^2 t)$
$\dot{\gamma}\,(t) = (-\sin t, 1, 2\sin t\cos t)$

$$\int_\gamma \omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^3 t + (\cos t + \sin^2 t) + (t + e^{\cos t})2\sin t\cos t =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t + \sin^2 t + 2t\sin t\cos t = 2\int_0^{\frac{\pi}{2}} \cos t + \sin^2 t + 2t\sin t\cos t\; dt =$$

$$= 2\sin t\Big|_0^{\frac{\pi}{2}} + \frac{\pi}{2} + 2\int_0^{\frac{\pi}{2}} t\sin 2t\; dt = 2 + \frac{\pi}{2} + 2\left(-\frac{1}{2}t\cos 2t\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2t\; dt\right) =$$

$$= 2 + \frac{\pi}{2} + 2\left(\frac{\pi}{4} + \frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{2}}\right) = 2 + \frac{\pi}{2} + \frac{\pi}{2} = 2 + \pi$$

Esercizio 3 Sia $\omega = \left(\dfrac{y^2e^x}{1+z^2} + 2x\sin^2 y\right)dx + \left(\dfrac{2ye^x}{1+z^2} + 2x^2\sin y\cos y\right)dy + \left(1 - \dfrac{2zy^2e^x}{(1+z^2)^2}\right)dz$

(a)
$$\frac{d}{dy}\omega_x = \frac{d}{dy}\left(\frac{y^2e^x}{1+z^2} + 2x\sin^2 y\right) = \frac{2ye^x}{1+z^2} + 4x\sin y\cos y$$
$$\frac{d}{dx}\omega_y = \frac{d}{dx}\left(\frac{2ye^x}{1+z^2} + 2x^2\sin y\cos y\right) = \frac{2ye^x}{1+z^2} + 4x\sin y\cos y$$
$$\frac{d}{dz}\omega_x = \frac{d}{dz}\left(\frac{y^2e^x}{1+z^2} + 2x\sin^2 y\right) = \frac{-2zy^2e^x}{(1+z^2)^2}$$
$$\frac{d}{dx}\omega_z = \frac{d}{dx}\left(1 - \frac{2zy^2e^x}{(1+z^2)^2}\right) = \frac{-2zy^2e^x}{(1+z^2)^2}$$
$$\frac{d}{dy}\omega_z = \frac{d}{dy}\left(1 - \frac{2zy^2e^x}{(1+z^2)^2}\right) = -\frac{4yze^x}{(1+z^2)^2}$$
$$\frac{d}{dz}\omega_y = \frac{d}{dz}\left(\frac{2ye^x}{1+z^2} + 2x^2\sin y\cos y\right) = -\frac{4yze^x}{(1+z^2)^2}$$
Siccome $\frac{d}{dy}\omega_x = \frac{d}{dx}\omega_y$,$\frac{d}{dx}\omega_z = \frac{d}{dz}\omega_x$ e $\frac{d}{dy}\omega_z = \frac{d}{dz}\omega_y$ allora ω è una forma chiusa

(b) Poichè ω è chiusa su tutto $\mathbb{R}^3$ allora ω è anche esatta. Cerchiamo una primitiva di ω cioè una funzione $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ tale che $\nabla f = (\omega_x, \omega_y, \omega_z)$.
$\dfrac{\partial f}{\partial z} = \left(1 - \dfrac{2zy^2e^x}{(1+z^2)^2}\right) \Longrightarrow f(x,y,z) = z + \dfrac{y^2e^x}{1+z^2} + c(x,y)$. Derivando in x otteniamo $\dfrac{\partial f}{\partial x} = \dfrac{y^2e^x}{1+z^2} + \dfrac{\partial c}{\partial x} = \omega_x = \dfrac{y^2e^x}{1+z^2} + 2x\sin^2 y$ da cui ricaviamo $\dfrac{\partial c}{\partial x} = 2x\sin^2 y \Longrightarrow c(x,y) = x^2\sin^2 y + k(y)$.
Quindi $f(x,y,z) = z + \dfrac{y^2e^x}{1+z^2} + x^2\sin^2 y + k(y)$. Infine imponendo che $\frac{\partial f}{\partial y} = \omega_y$ otteniamo che $\dfrac{2ye^x}{1+z^2} + 2x^2\sin y\cos y + k'(y) = \dfrac{2ye^x}{1+z^2} + 2x^2\sin y\cos y \Longrightarrow$

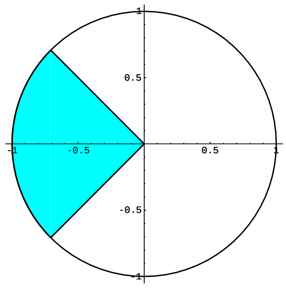
$k'(y) = 0 \Longrightarrow k(y) = k$ è una funzione costante. Dato che la primitiva di ω è definita a meno di costanti possiamo prendere $k = 0$.

Quindi una primitiva è $f(x, y, z) = z + \dfrac{y^2 e^x}{1 + z^2} + x^2 \sin^2 y$

(c) $\gamma(t) = \left(te^t \ , \ t\cos\left(\frac{\pi}{2}t\right), t^5\right) \quad t \in [0, 1]$

Siccome ω è esatta abbiamo che $\displaystyle\int_\gamma \omega = f(\gamma(1)) - f(\gamma(0)) = f(e, 0, 1) - f(0, 0, 0)$ $= 1 - 0 = 1$

Esercizio 5 $\omega = (x - y^2)dx - (x + y)dy \qquad D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \ , \ x + |y| \leq 0\}$



Dobbiamo verificare che $\displaystyle\int_{\partial^+ D} \omega = \int_D \frac{\partial \omega_y}{\partial x} - \frac{\partial w_x}{\partial y} dxdy$

$\displaystyle\int_D \frac{\partial \omega_y}{\partial x} - \frac{\partial w_x}{\partial y} dxdy = \int_D -1 + 2y \ dxdy =$ (passando in coordinate polari)

$$= \int_0^1 d\rho \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} d\rho \ 2\rho^2 \sin\theta - \rho = \int_0^1 d\rho \left(-2\rho^2 \cos\theta\Big|_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} - \frac{\pi}{2}\rho\right) = -\int_0^1 d\rho \ \frac{\pi}{2}\rho = -\frac{\pi}{4}$$

Calcoliamo ora il primo membro dell'uguaglianza. $\partial^+ D = \gamma_1 + \gamma_2 - \gamma_3$ dove

$\gamma_1(t) = (-t, t) \qquad t \in [0, \frac{1}{\sqrt{2}}]$

$\gamma_2(t) = (\cos t, \sin t) \qquad t \in [\frac{3}{4}\pi, \frac{5}{4}\pi]$

$\gamma_3(t) = (-t, -t) \qquad t \in [0, \frac{1}{\sqrt{2}}]$

Calcoliamo l'integrale di ω lungo questi tre cammini:

$\gamma_1(t) = (-t, t) \qquad t \in [0, \frac{1}{\sqrt{2}}] \qquad \dot{\gamma_1}\ (t) = (-1, 1)$

$$\int_{\gamma_1} \omega = \int_0^{\frac{1}{\sqrt{2}}} (-t - t^2)(-1) + 0dt = \int_0^{\frac{1}{\sqrt{2}}} t + t^2 \ dt = \frac{1}{2}t^2 + \frac{1}{3}t^3\Big|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{4} + \frac{1}{6\sqrt{2}}$$

$\gamma_2(t) = (\cos t, \sin t) \qquad t \in [\frac{3}{4}\pi, \frac{5}{4}\pi] \qquad \dot{\gamma_2}\ (t) = (-\sin t, \cos t)$

$$\int_{\gamma_2} \omega = \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} (\cos t - \sin^2 t)(-\sin t) - (\cos t + \sin t)\cos t \ dt =$$

$$= \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} -2\cos t \sin t + \sin^3 t - \cos^2 t \ dt = -\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \sin 2t \ dt + \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} (1 - \cos^2 t)\sin t \ dt$$

$$-\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \cos^2 t \ dt = \frac{1}{2}\cos 2t\Big|_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} + \int_{\frac{3}{4}\pi}^{\pi} (1 - \cos^2 t)\sin t \ dt + \int_{\pi}^{\frac{5}{4}\pi} (1 - \cos^2 t)\sin t \ dt +$$

$$-\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \cos^2 t \ dt = 0 + \int_{-1}^{-\frac{1}{\sqrt{2}}} (1 - t^2)dt - \int_{-1}^{-\frac{1}{\sqrt{2}}} (1 - t^2)dt - \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \cos^2 t \ dt =$$

$$= -\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \cos^2 t \ dt = -\int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \frac{1 + \cos(2t)}{2} \ dt = -\frac{\pi}{4} - \frac{1}{4}\sin(2t)\Big|_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} = -\frac{\pi}{4} - \frac{1}{2}$$

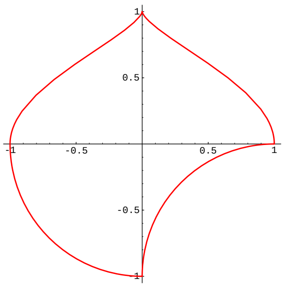
$\gamma_3(t) = (-t, -t) \qquad t \in [0, \frac{1}{\sqrt{2}}] \qquad \dot{\gamma_3}\ (t) = (-1, -1)$

$$\int_{\gamma_3} \omega = \int_0^{\frac{1}{\sqrt{2}}} t + t^2 - 2t \ dt = \int_0^{\frac{1}{\sqrt{2}}} t^2 - t \ dt = \frac{1}{3}t^3 - \frac{1}{2}t^2 \ \Big|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{6\sqrt{2}} - \frac{1}{4}$$

$$\int_{\partial^+ D} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega - \int_{\gamma_3} \omega = \frac{1}{4} + \frac{1}{6\sqrt{2}} - \frac{\pi}{4} - \frac{1}{2} - \frac{1}{6\sqrt{2}} + \frac{1}{4} = -\frac{\pi}{4}$$

Quindi $\int_{\partial^+ D} \omega = \int_D \frac{\partial \omega_y}{\partial x} - \frac{\partial w_x}{\partial y} dxdy$

Esercizio 7 Sia $\gamma(t) = \begin{cases} (\cos^3 t, \sin t) & \text{se } t \in [0, \pi] \\ (\cos t, \sin t) & \text{se } t \in [\pi, \frac{3}{2}\pi] \\ (1 + \sin t, -1 + \cos t) & \text{se } t \in [\frac{3}{2}\pi, 2\pi] \end{cases}$



Sia E la regione racchiusa dalla curva γ. Per il teorema di Gauss-Green si ha che $Area(E) = \int_\gamma xdy$. γ può essere divisa in tre curve $\gamma_1, \gamma_2, \gamma_3$:

$\gamma_1(t) = (\cos^3 t, \sin t) \quad t \in [0, \pi] \qquad \dot{\gamma_1}\ (t) = (-3\cos^2 t \sin t, \cos t)$

$\int_{\gamma_1} xdy = \int_0^\pi \cos^4 tt\ dt = \sin t \cos^3 t\Big|_0^\pi + 3\int_0^\pi \cos^2 t \sin^2 t\ dt = \frac{3}{4}\int_0^\pi \sin^2 2t\ dt =$

$= \frac{3}{8}\int_0^{2\pi} \sin^2 t\ dt = \frac{3}{8}\pi$

$\gamma_2(t) = (\cos t, \sin t) \quad t \in [\pi, \frac{3}{2}\pi] \qquad \dot{\gamma_2}\ (t) = (-\sin t, \cos t)$

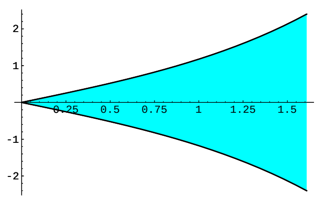
$\int_{\gamma_2} xdy = \int_\pi^{\frac{3}{2}\pi} \cos^2 t\ dt = \frac{\pi}{4}$

$\gamma_3(t) = (1 + \sin t, -1 + \cos t) \quad t \in [\frac{3}{2}\pi, 2\pi] \qquad \dot{\gamma_2}\ (t) = (\cos t, -\sin t)$

$\int_{\gamma_3} xdy = -\int_{\frac{3}{2}\pi}^{2\pi} (1 + \sin t)\sin t\ dt = -\int_{\frac{3}{2}\pi}^{2\pi} \sin t + \sin^2 t = \cos t\Big|_{\frac{3}{2}\pi}^{2\pi} - \frac{\pi}{4} =$

$= 1 - \frac{\pi}{4}$. Quindi $Area(E) = \int_\gamma xdy = \int_{\gamma_1} xdy + \int_{\gamma_2} xdy + \int_{\gamma_3} xdy = 1 + \frac{3}{8}\pi$

Esercizio 11 Siano $F(x,y) = (2y - x^2, ye^x)$ ed $E = \{(x,y) \in \mathbb{R}^2 \mid x \in [0, \log 2]\ ,\ |y| \le \sinh x\}$



Cominciamo con il calcolo diretto. $\partial E = \gamma_1\ \cup \gamma_2\ \cup\ \gamma_3$ dove

$\gamma_1(t) = (t, -\sinh t) \ \ t \in [0, \log 2] \qquad \gamma_2(t) = (\log 2, t) \ \ t \in [-\frac{3}{4}, \frac{3}{4}]$

$\gamma_3(t) = (t, \sinh t) \ t \in [0, \log 2]$

$\gamma_1(t) = (t, -\sinh t) \quad t \in [0, \log 2] \quad \dot{\gamma_1}\ (t) = (1, -\cosh t) \quad \tau(t) = \frac{(-\cosh t, -1)}{|\ \dot{\gamma_1}\ (t)|}$

$\int_{\gamma_1} < F, \tau > ds = \int_{\gamma_1} < F(\gamma(t)), \frac{(-\cosh t, -1)}{|\ \dot{\gamma_1}\ (t)|} > |\ \dot{\gamma_1}\ (t)|dt =$

$= \int_0^{\log 2} < (-2\sinh t - t^2, -\sinh t\; e^t), (-\cosh t, -1) > dt =$

$\int_0^{\log 2} 2\sinh t\cosh t + t^2\cosh t + \sinh t\; e^t\; dt = \sinh^2 t\Big|_0^{\log 2} + \int_0^{\log 2} t^2\cosh t + \frac{1}{2}e^{2t} - \frac{1}{2}\; dt =$

$= \frac{9}{16} + t^2\sinh t\Big|_0^{\log 2} - 2\int_0^{\log 2} t\sinh t\; dt + \frac{1}{4}e^{2t}\Big|_0^{\log 2} - \frac{1}{2}\log 2 = \frac{9}{16} + \frac{3}{4}\log^2 2 - 2t\cosh t\Big|_0^{\log 2} +$

$+2\int_0^{\log 2}\cosh t\; dt + 1 - \frac{1}{4} - \frac{1}{2}\log 2 = \frac{9}{16} + \frac{3}{4}\log^2 2 - \frac{5}{2}\log 2 + \frac{3}{2} + \frac{3}{4} - \frac{1}{2}\log 2 =$

$= \frac{45}{16} + \frac{3}{4}\log^2 2 - 3\log 2$

$\gamma_2(t) = (\log 2, t)\;\; t \in [-\frac{3}{4}, \frac{3}{4}] \qquad \dot{\gamma_2}\,(t) = (0,1) \qquad \tau(t) = \frac{(1,0)}{|\,\dot{\gamma_2}\,(t)|}$

$\int_{\gamma_2} < F, \tau > ds = \int_{-\frac{3}{4}}^{\frac{3}{4}} < F(\gamma_2(t)), \frac{(1,0)}{|\,\dot{\gamma_2}\,(t)|} > |\,\dot{\gamma_2}\,(t)|dt = \int_{-\frac{3}{4}}^{\frac{3}{4}} 2t - \log^2 2 = -\frac{3}{2}\log^2 2$

$\gamma_3(t) = (t, \sinh t)\;\; t \in [0, \log 2] \qquad \dot{\gamma_3}\,(t) = (1, \cosh t) \qquad \tau(t) = \frac{(-\cosh t, 1)}{|\,\dot{\gamma_3}\,(t)|}$

$\int_{\gamma_3} < F, \tau > ds = \int_0^{\log 2} < F(\gamma_3(t)), \frac{(-\cosh t, 1)}{|\,\dot{\gamma_1}\,(t)|} > |\,\dot{\gamma_1}\,(t)|dt =$

$= \int_0^{\log 2} < (2\sinh t - t^2, \sinh t\; e^t), (-\cosh t, 1) > dt = \int_0^{\log 2} -2\sinh t\cosh t + t^2\cosh t + e^t\sinh t dt$

$= -\sinh^2 t\Big|_0^{\log 2} + \int_0^{\log 2} t^2\cosh t + \frac{1}{2}\int_0^{\log 2} e^{2t} - 1\; dt = -\frac{9}{16} + t^2\sinh t\Big|_0^{\log 2} - \int_0^{\log 2} 2t\sinh t +$

$+\frac{1}{4}e^{2t}\Big|_0^{\log 2} - \frac{1}{2}\log 2 = -\frac{9}{16} + \frac{3}{4}\log^2 2 - 2t\cosh t\Big|_0^{\log 2} + 2\int_0^{\log 2}\cosh t\; dt + 1 - \frac{1}{4} - \frac{1}{2}\log 2 =$

$= -\frac{9}{16} + \frac{3}{4}\log^2 2 - \frac{5}{2}\log 2 + \frac{3}{2} + \frac{3}{4} - \frac{1}{2}\log 2 = -\frac{9}{16} + \frac{9}{4} + \frac{3}{4}\log^2 2 - 3\log 2 =$

$= \frac{27}{16} + \frac{3}{4}\log^2 2 - 3\log 2$

Quindi $\int_{\partial E} < F, \tau > \; ds = \int_{\gamma_1} < F, \tau > + \int_{\gamma_2} < F, \tau > + \int_{\gamma_3} < F, \tau >= \frac{9}{2} - 6\log 2$

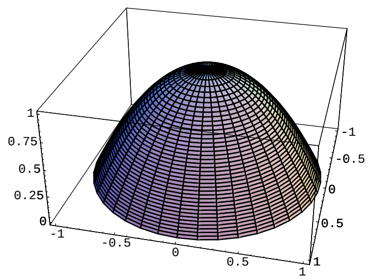
Calcoliamo ora lo stesso integrale utilizzando il teorema della divergenza:

$\int_{\partial E} < F, \tau > ds = \int_E div\; F dxdy = \int_E -2x + e^x\; dxdy = \int_0^{\log 2} dx \int_{-\sinh x}^{\sinh x} dy\; e^x - 2x =$

$= \int_0^{\log 2} dx\; 2e^x\sinh x - 4x\sinh x = \int_0^{\log 2} e^{2x} - 1 + 4x\sinh x = \frac{1}{2}e^{2x}\Big|_0^{\log 2} - \log 2 +$

$-4x\cosh x\Big|_0^{\log 2} + 4\int_0^{\log 2} dx\;\; \cosh x = 2 - \frac{1}{2} - \log 2 - 5\log 2 + 3 = \frac{9}{2} - 6\log 2$

Esercizio 13 Sia $F(x,y,z) = \big(x - y\;,\; xyz\;,\; xy^3\big)$. Verifichiamo la validità del teorema della divergenza nell' insieme $E = \{(x,y,z) \in \mathbb{R}^3 \mid 0 \le z \le 1 - x^2 - y^2\}$.



Dobbiamo verificare che $\int_E divF dxdydz = \int_{\partial E} F \cdot \nu \, d\sigma$

$divF = \frac{\partial F_x}{\partial x} + \frac{D\ F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 + xz$

$$\int_E divF\ dxdydz = \int_E 1 + xz\ dxdydz = \int_{x^2+y^2\le 1} dxdy \int_0^{1-x^2-y^2} dz\ 1 + xz =$$

$$= \int_{\{x^2+y^2\le 1\}} dz\ z + \frac{1}{2}xz^2\Big|_0^{1-x^2-y^2} = \int_{\{x^2+y^2\le 1\}} dz\ 1 - x^2 - y^2 + x(1-x^2-y^2) =$$

$$= \int_0^1 d\rho \int_0^{2\pi} d\theta\ \rho(1-\rho^2+\rho\cos\theta(1-\rho^2)) = \int_0^1 d\rho \int_0^{2\pi} d\theta \rho - \rho^3 - \rho^2\cos\theta(1-\rho^2)$$

$$\int_0^1 d\rho \int_0^{2\pi} d\theta\ \rho - \rho^3 = 2\pi \int_0^1 d\rho\ \rho - \rho^3 = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{2}$$

Calcoliamo ora $\int_{\partial E} F \cdot \nu \, d\sigma$. $\partial E = \Sigma_1 + \Sigma_2$ con

$\Sigma_1 = \{(x,y,z) \mid z = 1 - x^2 - y^2\ ,\ x^2+y^2 \le 1\}$ e $\Sigma_2 = \{(x,y,0) \mid x^2+y^2 \le 1\}$

Una parametrizzazione di Σ_1 è $\Phi(u,v) = (u, v, 1-u^2-v^2)$ con $u^2+v^2 \le 1$.

$\Phi_u = (1, 0, -2u)$

$\Phi_v = (0, 1, -2v)$

$\Phi_u \wedge \Phi_v = (2u, 2v, 1)$ quindi $\nu(u,v) = \frac{\Phi_u \wedge \Phi_v}{|\Phi_u \wedge \Phi_v|}$

$$\int_{\Sigma_1} F \cdot \nu\, d\sigma = \int_{\{u^2+v^2\le 1\}} (u-v, uv(1-u^2-v^2), uv^3) \cdot \frac{(2u,2v,1)}{|\Phi_u \wedge \Phi_v|} |\Phi_u \wedge \Phi_v| dudv =$$

$$= \int_{\{u^2+v^2\le 1\}} 2u^2 - 2uv + 2uv^2(1-u^2-v^2) + uv^3\ dudv = \int_{\{u^2+v^2\le 1\}} 2u^2\ dudv =$$

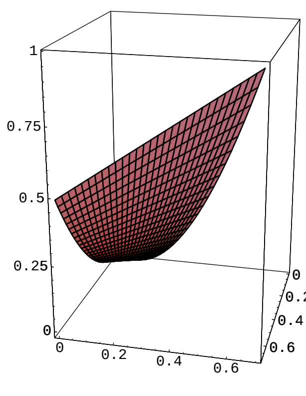
$$= 2\int_0^1 d\rho \int_0^{2\pi} d\theta\ \rho^3\cos^2\theta = 2\pi \int_0^1 \rho^3\ d\rho = \frac{\pi}{2}.$$

Infine una parametrizzazione di Σ_2 è $\Phi(u,v) = (u,v,0)$.

$$\int_{\Sigma_2} F \cdot \nu\, d\sigma = \int_{\Sigma_2} F(x,y,z)\cdot(0,0,-1) d\sigma = \int_{\Sigma_2} xy^3 d\sigma = \int_{\{u^2+v^2\le 1\}} uv^3 dudv = 0$$

Quindi $\int_{\partial E} F \cdot \nu\, d\sigma \int_{\Sigma_1} F \cdot \nu\, d\sigma + \int_{\Sigma_2} F \cdot \nu\, d\sigma = \frac{\pi}{2}$. Pertanto vale il teorema delle divergenza $\int_{\partial E} F \cdot \nu\, d\sigma = \int_E divF dxdydz = \frac{\pi}{2}$

Esercizio 17 $\omega = xz\ dx + y\ dy + \left(\frac{1}{2}x^2 + yz\right) dz$. Verifichiamo la validità del teorema di Stokes per ω sulla superficie $z = x^2 + xy$ con $(x,y) \in D = \{(x,y) \mid 0 \le x \le 1\ ,\ 0 \le y \le x\}$



Dobbiamo provare che $\displaystyle\int_{\partial^+\Sigma} \omega = \int_\Sigma rot\ F \cdot \nu\ d\sigma$

Una parametrizzazione di Σ è $\Phi(u,v) = (u, v, u^2 + uv)$

$\Phi_u = (1, 0, 2u + v)$

$\Phi_v = (0, 1, u)$

$\Phi_u \wedge \Phi_v = (-2u - v, -u, 1)$

$$rot\ F = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & y & \frac{1}{2}x^2 + yz \end{vmatrix} = (z, 0, 0)$$

$$\int_\Sigma rot\ F \cdot \nu\ d\sigma = \int_D rot\ F(\Phi(u,v)) \cdot \frac{\Phi_u \wedge \Phi_v}{|\Phi_u \wedge \Phi_v|} |\Phi_u \wedge \Phi_v| dudv =$$

$$= \int_D < (u^2 + uv, 0, 0), (-2u - v, -u, 1) > dudv = \int_D -2u^3 - 3u^2v - uv^2\ dudv =$$

$$= \int_0^1 du \int_0^u dv\ -2u^3 - 3u^2v - uv^2\ dudv = \int_0^1 du\ -2u^3v - \frac{3}{2}u^2v^2 - \frac{1}{3}uv^3\Big|_0^u =$$

$$= \int_0^1 -2u^4 - \frac{3}{2}u^4 - \frac{1}{3}u^4\ du = \int_0^1 du\ -\frac{23}{6}u^4\ du = -\frac{23}{30}$$

Calcoliamo ora l'altro membro dell' uguaglianza.

$\partial D = \alpha_1 + \alpha_2 - \alpha_3$ dove $\alpha_1(t) = (t, 0)$ $t \in [0,1]$ $\alpha_2(t) = (1, t)$ $t \in [0,1]$

$\alpha_3(t) = (t, t)$ $t \in [0,1]$

Quindi $\partial^+\Sigma = \gamma_1 + \gamma_2 - \gamma_3$ dove

$\gamma_1(t) = \Phi(\alpha_1(t)) = (t, 0, t^2)$ $t \in [0,1]$

$\gamma_2(t) = \Phi(\alpha_2(t)) = (1, t, 1 + t)$ $t \in [0,1]$

$\gamma_3(t) = \Phi(\alpha_3(t)) = (t, t, 2t^2)$ $t \in [0,1]$

$$\int_{\gamma_1} \omega = \int_0^1 < F(t, 0, t^2), \dot{\gamma_1}(t) > dt = \int_0^1 < (t^3, 0, \frac{1}{2}t^2), (1, 0, 2t) > dt = \int_0^1 2t^3\ dt = \frac{1}{2}$$

$$\int_{\gamma_2} \omega = \int_0^1 < F(1, t, 1 + t), \dot{\gamma_2}(t) > dt = \int_0^1 < (1 + t, t, \frac{1}{2} + t + t^2), (0, 1, 1) > dt =$$

$$= \int_0^1 t + \frac{1}{2} + t + t^2\ dt = \int_0^1 \frac{1}{2} + 2t + t^2\ dt = \frac{1}{2} + 1 + \frac{1}{3} = \frac{11}{6}$$

$$\int_{\gamma_3} \omega = \int_0^1 < F(t, t, 2t^2), \dot{\gamma_1}(t) > dt = \int_0^1 < (2t^3, t, \frac{1}{2}t^2 + 2t^3), (1, 1, 4t) > dt =$$

$$= \int_0^1 2t^3 + t + 2t^3 + 8t^4\ dt = \int_0^1 t + 4t^3 + 8t^4\ dt = \frac{1}{2} + 1 + \frac{8}{5} = \frac{31}{10}$$

Quindi $\displaystyle\int_{\partial^+\Sigma} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega - \int_{\gamma_3} \omega = \frac{1}{2} + \frac{11}{6} - \frac{31}{10} = -\frac{23}{30}$

Esercizio 19 Sia $\displaystyle\omega = \left(-\frac{8e^{x^2}x}{(4x^2 + y^2)^2} + \frac{2e^{x^2}}{4x^2 + y^2}\right) dx - 2\left(y + \frac{e^{x^2}y}{(4x^2 + y^2)^2}\right) dy.$

(a) Calcolare $\displaystyle\int_\gamma \omega$ dove γ è il bordo dell'ellisse $x^2 + \dfrac{y^2}{4} \leq 1$

$\gamma(t) = (\cos t, 2\sin t)$ $t \in [0, 2\pi]$

$\dot{\gamma}(t) = (-\sin t, 2\cos t)$

$$\int_\gamma \omega = \int_0^{2\pi} \frac{8e^{\cos^2 t}\cos t}{16}\sin t - \frac{2e^{\cos^2 t}}{16}\cos t \sin t - 8\sin t\cos t - 2\frac{2e^{\cos^2 t}\sin t}{16} 2\cos t\ dt =$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{\cos^2 t}\sin t - 4\sin(2t)dt = -\frac{1}{2}\int_0^{2\pi} e^{\cos^2 t}\sin t\ dt = -\frac{1}{2}\int_0^{\pi} e^{\cos^2 t}\sin t\ dt +$$

$$-\frac{1}{2}\int_\pi^{2\pi} e^{\cos^2 t}\sin t\ dt = -\frac{1}{2}\int_{-1}^1 e^{t^2}dt + \frac{1}{2}\int_{-1}^1 e^{t^2}dt = 0$$

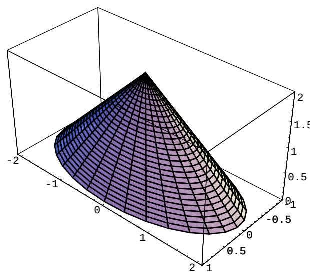
(b) $\dfrac{d}{dy}\omega_x = \dfrac{32xye^{x^2}}{(4x^2+y^2)^3} - \dfrac{4xye^{x^2}}{(4x^2+y^2)^2}$

$\dfrac{d}{dx}\omega_y = -2\left(\dfrac{2xye^{x^2}(4x^2+y^2)^2 - 16xye^{x^2}(4x^2+y^2)}{(4x^2+y^2)^4}\right) = \dfrac{32xye^{x^2}}{(4x^2+y^2)^3} - \dfrac{4xye^{x^2}}{(4x^2+y^2)^2}$

Quindi ω è chiusa.

(c) Proviamo che se $\alpha : I \longrightarrow \mathbb{R}^2 - \{(0,0)\}$ è una qualsiasi curva chiusa allora $\int_\alpha \omega = 0$. Possiamo supporre che α sia una curva semplice (perchè ogni curva chiusa può essere decomposta in numero finito di curve semplici). Se α non gira attorno all'origine allora sicuramente $\int_\alpha \omega = 0$ perchè la regione chiusa racchiusa da α è una regione semplicemente connessa di $\mathbb{R}^2$ in cui ω è chiusa e quindi esatta. Supponiamo dunque che α sia una curva che gira attorno all'origine. Possiamo supporre che α sia tutta contenuta nell'ellisse F di equazione $x^2 + \frac{y^2}{4} \leq 1$ oppure che rimanga sempre all'esterno di F (Infatti se questo non è vero allora α può essere divisa in una famiglia di curve semplici che o sono contenute in F oppure rimangono all'esterno di F). Supponiamo prima che α sia tutta contenuta in F; chiamiamo E la regione delimitata da α e dal bordo γ dell'ellisse.Siccome ω è chiusa allora per il teorema di Gauss-Green $\int_{\partial^+ E} \omega = \int_E \frac{\partial}{\partial x}\omega_y - \frac{\partial}{\partial y}\omega_x \, dxdy = 0$. Ma $\partial^+ E = \gamma - \alpha$ e quindi $0 = \int_{\partial^+(F-E)} \omega = \int_\gamma \omega - \int_\alpha \omega = -\int_\alpha \omega \Longrightarrow \int_\alpha \omega = 0$. Allo stesso modo se α rimane all'esterno di F e indichiamo con E la regione racchiusa da α e γ utilizzando il teorema di Gauss-Green si ricava che $\int_\alpha \omega = 0$. Quindi ω è esatta in $\mathbb{R}^2 - \{(0,0)\}$ perchè l'integrale di α su ogni curva chiusa è 0.

Esercizio 23 Calcoliamo $\int_{\partial C} F \cdot \nu \, d\sigma$ dove C è il cono avente per base l' ellisse $4x^2 + y^2 \leq 4$ del piano $z = 0$ con vertice nel punto $(0,0,2)$ e $F(x,y,z) = (x^2 z, yz + x^3, x - yz^2)$.
$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + \dfrac{y^2}{4} = \left(1 - \dfrac{z}{2}\right)^2 \ , \ 0 \leq z \leq 2\}$



Cominciamo con il calcolo diretto $\partial C = \Sigma_1 \cup \Sigma_2$ dove $\Sigma_1 = \{(x,y,0) \mid x^2 + \frac{y^2}{4} \leq 1\}$
$\Sigma_2 = \{(x,y,z) \mid x^2 + \frac{y^2}{4} = (1 - \frac{z}{2})^2 \ , \ 0 \leq z \leq 2\}$
Una parametrizzazione di Σ_1 è
$\Phi(u,v) = (u,v,0)$ con $(u,v) \in E = \{(x,y) \mid x^2 + \frac{y^2}{4} \leq 1\}$
$\Phi_u = (1,0,0) \quad \Phi_v = (0,1,0) \quad \Phi_u \wedge \Phi_v = (0,0,1) \quad$ e $\nu(u,v) = (0,0,-1)$

$$\int_{\Sigma_1} F \cdot \nu d\sigma = \int_E < F(u,v,0), -\frac{\Phi_u \wedge \Phi_v}{|\Phi_u \wedge \Phi_v|} > |\Phi_u \wedge \Phi_v| dudv =$$
$$= \int_E < F(u,v,0), (0,0,-1) > dudv = \int_E u \, dudv = 0.$$

Una parametrizzazione di Σ_2 è $\Phi(u,v) = (u\cos v, 2u\sin v, 2-2u)$ con $u \in [0,1]$ e $v \in [0,2\pi]$

$\Phi_u = (\cos v, 2\sin v, -2)$ $\Phi_v = (-u\sin v, 2u\cos v, 0)$

$\Phi_u \wedge \Phi_v = (4u\cos v, 2u\sin v, 2u)$

$$\int_{\Sigma_2} F\cdot\nu d\sigma = \int_0^1 du \int_0^{2\pi} dv\ F(u\cos v, 2u\sin v, 2-2u)\cdot\frac{\Phi_u\wedge\Phi_v}{|\Phi_u\wedge\Phi_v|}|\Phi_u\wedge\Phi_v|dudv =$$

$$= \int_0^1 du \int_0^{2\pi} dv\ (u^2\cos^2 v\ (2-2u), 2u(2-2u)\sin v + u^3\cos^3 v, u\cos v - 2u(2-2u)^2\sin v).(\Phi_u\wedge\Phi_v)$$

$$= \int_0^1 du \int_0^{2\pi} dv\ 4u^3(2-2u)\cos^3 v + 4u^2(2-2u)\sin^2 v + 2u^4\cos^3 v\sin v + 2u^2\cos v - u^2(2-2u)^2\sin v$$

$$= 4\int_0^1 du \int_0^{2\pi} dv\ u^2(2-2u)\sin^2 v = 4\pi\int_0^1 u^2(2-2u)\ du = 8\pi\int_0^1\ u^2 - u^3\ du =$$

$$= 8\pi\left(\frac{1}{3}u^3 - \frac{1}{4}u^4\Big|_0^1\right) = 8\pi\left(\frac{1}{3}-\frac{1}{4}\right) = 8\pi\ \frac{1}{12} = \frac{2}{3}\pi$$

Quindi $\int_{\partial E} F\cdot\nu\ d\sigma = \int_{\Sigma_1} F\cdot\nu\ d\sigma + \int_{\Sigma_2} F\cdot\nu\ d\sigma = 0 + \frac{2}{3}\pi = \frac{2}{3}\pi$

$div\ F = 2xz + z - 2yz$

Facciamo ora il calcolo utilizzando il teorema della divergenza:

$$\int_E div\ Fdxdydz = \int_0^2 dz\int_{E_z} div\ F\ dxdydz = \int_0^2 dz\int_{E_z} 2xz + z - 2yz\ sxdy.$$

Fissata z E_z è una ellisse di equazione $x^2 + \frac{y^2}{4} = \left(1-\frac{z}{2}\right)^2$ cioè una ellisse con semiassi di lunghezza $1-\frac{z}{2}$ e $2-z$ quindi per integrare su E_z possiamo passare in coordinate ellittiche

$(x,y) = ((1-\frac{z}{2})\rho\cos\theta, (2-z)\rho\sin\theta)$ con $\rho\in[0,1]$ e $\theta\in[0,2\pi]$

$$\int_0^2 dz\int_{E_z} 2xz + z - 2yz\ dxdy = \int_0^2 dz\ z\int_{E_z} 2x + 1 - 2y\ dxdy =$$

$$= \int_0^2 dz\ z\int_0^1 d\rho\int_0^{2\pi} d\theta((1-\frac{z}{2})\rho\cos\theta + 1 + (2-z)\rho\sin\theta)(2-z)(1-\frac{z}{2})\rho =$$

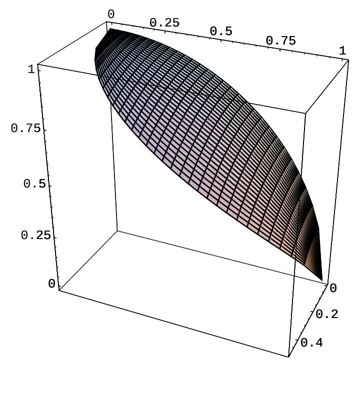
$$= \frac{1}{2}\int_0^2 dz\ z(2-z)^2\int_0^1 d\rho\int_0^{2\pi} d\theta((1-\frac{z}{2})\rho\cos\theta + 1 + (2-z)\rho\sin\theta)\rho =$$

$$= \pi\int_0^2 dz\ z(2-z)^2\int_0^1 d\rho\ \rho = \frac{\pi}{2}\int_0^2 z(2-z)^2 dz = \frac{\pi}{2}\int_0^2 dz\ 4z - 4z^2 + z^3 =$$

$$= \frac{\pi}{2}\left(2z^2 - \frac{4}{3}z^3 + \frac{1}{4}z^4\Big|_0^2\right) = \frac{\pi}{2}\left(8 - \frac{32}{3} + 4\right) = \frac{2}{3}\pi$$

Esercizio 29 $\Sigma = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 = 1\ ,\ 0\le x\le\sqrt{y-y^2}\ ,\ z\ge 0\}$ e

$$\omega = (x-y)dx + \frac{yz^2}{x^2+y^2+z^2}dy + x^2z\ dz.$$



Una parametrizzazione di Σ è
$\Phi(u,v) = (u, v, \sqrt{1-u^2-v^2})$ con $(u,v) \in H = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \sqrt{y-y^2}\ \}$
$\Phi_u = (1, 0, -\dfrac{u}{\sqrt{1-u^2-v^2}}) \quad \Phi_v = (0, 1, -\dfrac{v}{\sqrt{1-u^2-v^2}})$
$\Phi_u \wedge \Phi_v = (\dfrac{u}{\sqrt{1-u^2-v^2}}, \dfrac{v}{\sqrt{1-u^2-v^2}}, 1)$
$|\Phi_u \wedge \Phi_v| = \sqrt{\dfrac{u^2+v^2}{1-u^2-v^2}+1} = \dfrac{1}{\sqrt{1-u^2-v^2}}$
Sia $F(x,y,z) = (x-y, \frac{yz^2}{x^2+y^2+z^2}, x^2 z)$

$$rotF(x,y,z) = \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x-y & \frac{yz^2}{x^2+y^2+z^2} & x^2 z \end{vmatrix} =$$

$$= \left(-\frac{2yz}{x^2+y^2+z^2} + \frac{2yz^3}{(x^2+y^2+z^2)^2}, -2xz, -\frac{2xyz^2}{(x^2+y^2+z^2)} + 1\right)$$

$$\int_\Sigma < rotF\ ,\ \nu > d\sigma = \int_H < rotF(\Phi(u,v))\ ,\ \frac{\Phi_u \wedge \Phi_v}{|\Phi_u \wedge \Phi_v|} > |\Phi_u \wedge \Phi_v| dudv =$$

$$\int_H < (-2v\sqrt{1-u^2-v^2} + 2v\sqrt{(1-u^2-v^2)^3}, -2u\sqrt{1-u^2-v^2}, 1 - 2uv(1-u^2-v^2)), \Phi_u \wedge \Phi_v >=$$

$$= \int_H -2uv + 2uv(1-u^2-v^2) - 2uv - 2uv(1-u^2-v^2) + 1\ dudv =$$

$$= \int_H 1 - 4uv\ dudv = \int_0^1 dv \int_0^{\sqrt{v-v^2}} du\ 1 - 4uv = \int_0^1 dv\ u - 2u^2 v\Big|_0^{\sqrt{v-v^2}} =$$

$$= \int_0^1 dv\ \sqrt{v-v^2} - 2(v-v^2)v = \int_0^1 dv\ \sqrt{v-v^2} - 2v^2 + 2v^3\ dv = \int_0^1 \sqrt{v(1-v)} dv +$$

$$= -\frac{2}{3} + \frac{1}{2} = -\frac{1}{6} + \int_0^1 \sqrt{v(1-v)} \overset{(v=\sin^2 t)}{=} -\frac{1}{6} + \int_0^{\frac{\pi}{2}} 2\sin t \cos t \sqrt{\sin^2 t \cos^2 t}\ dt =$$

$$= -\frac{1}{6} + 2\int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t\ dt = -\frac{1}{6} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^2 2t\ dt = -\frac{1}{6} + \frac{1}{4}\int_0^{\pi} \sin^2 t = -\frac{1}{6} + \frac{\pi}{8}$$

$D^+H = \alpha_1 - \alpha_2$ dove
$\alpha_1(t) = (\frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t) \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ e $\alpha_2(t) = (0, t) \quad t \in [0, 1]$
$\partial^+\Sigma = \gamma_1 - \gamma_2$ dove
$\gamma_1(t) = \Phi(\alpha_1(t)) = (\frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t, \sqrt{1 - \frac{1}{4}\cos^2 t - \frac{1}{4}\sin^2 t - \frac{1}{4} - \frac{1}{2}\sin t}) =$
$= (\frac{1}{2}\cos t, \frac{1}{2} + \frac{1}{2}\sin t, \frac{1}{\sqrt{2}}\sqrt{1 - \sin t}) \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
$\gamma_2(t) = (0, t, \sqrt{1-t^2}) \quad t \in [0, 1]$

$$\int_{\gamma_2} \omega = \int_0^1 < (-t, t - t^3, 0), (0, 1, \frac{1}{\sqrt{1-t^2}}) > dt = \int_0^1 t - t^3\ dt = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\int_{\gamma_1} \omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} < (\frac{1}{2}\cos t - \frac{1}{2} - \frac{1}{2}\sin t, (\frac{1}{2} + \frac{1}{2}\sin t)\frac{1}{2}(1 - \sin t), \frac{1}{4\sqrt{2}}\cos^2 t\sqrt{1-\sin t}, \dot{\gamma_1} > dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\frac{1}{4}\sin t \cos t + \frac{1}{4}\sin t + \frac{1}{4}\sin^2 t + \frac{1}{8}(1 - \sin^2 t)\cos t - \frac{1}{16}\cos^3 t\ dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\sin^2 t + \frac{1}{8}\cos t - \frac{1}{8}\sin^2 t \cos t - \frac{1}{16}\cos^3 t = \frac{\pi}{8} + \frac{1}{8}\sin t - \frac{1}{24}\sin^3 t\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} +$$

$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 t\ dt = \frac{\pi}{8} + \frac{1}{4} - \frac{1}{12} - \frac{1}{8}\int_0^{\frac{\pi}{2}} (1 - \sin^2 t)\cos t = \frac{\pi}{8} + \frac{1}{4} - \frac{1}{12} - \frac{1}{8}\int_0^1 (1 - s^2) ds$$

$$= \frac{\pi}{8} + \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \frac{\pi}{8} + \frac{1}{4} - \frac{1}{6}.$$

Dunque $\displaystyle\int_{D^+\Sigma} \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \frac{\pi}{8} - \frac{1}{6}$