

## AM2: Tracce delle lezioni-V settimana

### SERIE DI POTENZE NEL CAMPO COMPLESSO

**1. Definizione**  $z_n \in \mathbf{C}$  converge a  $z$  ( $z_n \rightarrow_n z$ )  $\Leftrightarrow |z_n - z| \rightarrow_n 0$ .

Siccome  $|z_n - z|^2 = |\operatorname{Re}z_n - \operatorname{Re}z|^2 + |\operatorname{Im}z_n - \operatorname{Im}z|^2$ , si ha che:

$$z_n \rightarrow_n z \Leftrightarrow \operatorname{Re}z_n \rightarrow_n \operatorname{Re}z \quad \text{e} \quad \operatorname{Im}z_n \rightarrow_n \operatorname{Im}z$$

$$z_n \rightarrow_n z \Leftrightarrow \forall \epsilon > 0, \exists n_\epsilon : n, m \geq n_\epsilon \Rightarrow |z_n - z_m| \leq \epsilon \quad (\mathbf{Cauchy})$$

**2. Definizione**  $\sum_{n=1}^{\infty} z_n$  converge sse  $S_N := \sum_{n=1}^N z_n$  converge.

$\sum_n z_n$  si dice assolutamente convergente se  $\sum_n |z_n| < +\infty$ .

$$(\mathbf{Cauchy}) \quad \sum_n z_n \text{ converge} \Leftrightarrow \forall \epsilon > 0, \exists N_\epsilon : \left| \sum_{n=N}^{N+p} z_n \right| \leq \epsilon \quad \forall N \geq N_\epsilon, \forall p.$$

In particolare,  $\sum_n |z_n| < +\infty \Rightarrow \sum_n z_n$  converge e in particolare,

$$\limsup_n |z_n|^{\frac{1}{n}} < 1 \Rightarrow \sum_n |z_n| < +\infty \Rightarrow \sum_n z_n \text{ converge.} \quad \text{Si ha cos\`i}$$

**3. Cauchy-Hadamard** Sia  $a_n \in \mathbf{C}$ ,  $r := \limsup_n |a_n|^{-\frac{1}{n}}$ . Allora

$$z \in \mathbf{C}, \quad |z| < r \Rightarrow \sum_{n=0}^{\infty} |a_n z^n| < +\infty, \quad |z| > r \Rightarrow \sum_{n=0}^{\infty} |a_n z^n| = +\infty$$

$r :=$  raggio di convergenza,  $D_r := \{z : |z| < r\} :=$  disco di convergenza.

$$\text{ESEMPLI.} \quad \sum_{n=0}^{\infty} z^n \text{ converge in } |z| < 1 \quad \text{e} \quad \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

$$\exp z := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{converge} \quad \forall z \in \mathbf{C}$$

**4. Definizione**  $O \subset \mathbf{C}$  \u00e9 aperto, se  $z_0 \in O, z_n \rightarrow_n z_0 \Rightarrow z_n \in O$  definitivamente (ovvero  $z_0 \in O \Rightarrow \exists D_r(z_0) := \{z \in \mathbf{C} : |z - z_0| < r\} \subset O$ ).

**5. Funzioni complesse di variabile complessa.** Sia  $f : O \rightarrow \mathbf{C}$ .

$f$  \u00e9 continua in  $z_0 \in O \Leftrightarrow \forall \epsilon > 0 \exists \delta > 0 : |z - z_0| \leq \delta \Rightarrow |f(z) - f(z_0)| \leq \epsilon$

$f$  \u00e9 derivabile in  $z_0 \in O$  con derivata  $f'(z_0)$   $\Leftrightarrow$

$$\forall \epsilon > 0 \exists \delta > 0 : |z - z_0| \leq \delta \Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| \leq \epsilon$$

Anche qui, come nel caso reale:  $f$  é derivabile in  $z_0 \Rightarrow f$  é continua in  $z_0$ .

**6. Esempio** Sia  $f(z) := \sum_{n=0}^{\infty} a_n z^n$  somma di una serie di potenze avente raggio di convergenza  $r > 0$ . Allora  $f \in C^\infty(D_r)$ .

Proviamo che  $\frac{d^n f}{dz^n}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{n!} a_{n+k} z^k \quad \forall z \in D_r$ . Come nel caso reale, basta provare la formula per  $n = 1$ . Siano  $z, z_0 \in D_\rho, \rho < r$ . É

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - \sum_{n=1}^{\infty} n a_n z_0^{n-1} \right| \leq \sum_{n=2}^{\infty} |a_n| \left| \frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right|$$

Ora,  $|z^n - z_0^n| = |(z - z_0)(z^{n-1} + z_0 z^{n-2} + \dots + z_0^{n-2} z + z_0^{n-1})| \leq |z - z_0| n \rho^{n-1} \Rightarrow$

$$\begin{aligned} \left| \frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right| &= |z^{n-1} - z_0^{n-1} + z_0(z^{n-2} - z_0^{n-2}) + \dots + z_0^{n-2}(z - z_0)| \leq \\ &\leq |z^{n-1} - z_0^{n-1}| + |z_0| |z^{n-2} - z_0^{n-2}| + \dots + |z_0|^{n-2} |z - z_0| \leq \\ &\leq |z - z_0| \left[ (n-1)\rho^{n-2} + (n-2)|z_0|\rho^{n-3} + \dots + |z_0|^{n-2} \right] \leq \frac{n(n-1)}{2} |z - z_0| \rho^{n-2} \\ &\Rightarrow \sum_{n=2}^{\infty} |a_n| \left| \frac{z^n - z_0^n}{z - z_0} - n z_0^{n-1} \right| \leq |z - z_0| \sum_{n=2}^{\infty} |a_n| \frac{n(n-1)}{2} \rho^{n-2} \xrightarrow{z \rightarrow z_0} 0 \end{aligned}$$

perché  $\rho < r \Rightarrow \sum_{n=2}^{\infty} |a_n| \frac{n(n-1)}{2} \rho^{n-2} < +\infty$ .

**7.**  $\exp(z+w) = \exp z + \exp w \quad \forall z, w \in \mathbf{C}$ . Segue da

**8. Lemma (Prodotto secondo Cauchy).**  $\sum_{n=0}^{\infty} |z_n| + \sum_{n=0}^{\infty} |w_n| < +\infty \Rightarrow$

$$\sum_{n=0}^{\infty} \left| \sum_{j+k=n} z_j w_k \right| < +\infty, \quad \sum_{n=0}^{\infty} \left( \sum_{j+k=n} z_j w_k \right) = \left( \sum_{n=0}^{\infty} z_n \right) \left( \sum_{n=0}^{\infty} w_n \right)$$

Infatti,  $\exp(z+w) = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{1}{n!} \sum_{j+k=n} \frac{n!}{j!k!} z^j w^k \right) =$

$$\sum_{n=0}^{\infty} \left( \sum_{j+k=n} \frac{z^j w^k}{j!k!} \right) = \left( \sum_{n=0}^{\infty} \frac{z^j}{j!} \right) \left( \sum_{n=0}^{\infty} \frac{w^k}{k!} \right) = \exp z \exp w$$

In particolare,  $(\exp z)^p = \exp(pz) \quad \forall p \in \mathbf{N}, z \in \mathbf{C}, \exp p = (\exp 1)^p =: e^p, (\exp(\frac{1}{p}))^p = e$ . Dunque  $\exp(\frac{1}{p}) = e^{\frac{1}{p}}$  e quindi  $\exp(\frac{p}{q}) = (\exp \frac{1}{q})^p = (e^{\frac{1}{q}})^p = e^{\frac{p}{q}}: x \rightarrow \exp x, x \in \mathbf{R}$  é prolungamento continuo di  $r \rightarrow e^r, r \in \mathbf{Q}$ .

**9. Formule di Eulero**

$$\exp(\pm it) = \cos t \pm i \sin t \quad \forall t \in \mathbf{R}$$

$$\sin t = \frac{\exp(it) - \exp(-it)}{2i}, \quad \cos t = \frac{\exp(it) + \exp(-it)}{2} \quad \forall t \in \mathbf{R}$$

Da  $\exp(it) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}$  segue  $Re(\exp it) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} = \cos t$ ,  $Im(\exp it) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = \sin t$ . In particolare,  $\exp(2k\pi i) = 1 \quad \forall k \in \mathbf{Z}$ , cioè  $t \rightarrow \exp(it)$   $t \in \mathbf{R}$  é  $2\pi$ -periodica.

Infine, da 7.,  $\exp(x + iy) = \exp x \exp(iy) = e^x (\cos t + i \sin t)$ . In particolare,  $\exp(z + 2\pi i) = \exp z$ ,  $\forall z$ .

**10.** (i)  $\exp(-z) = (\exp z)^{-1}$  (ii)  $\overline{\exp z} = \exp \bar{z}$

(iii)  $|\exp(it)| = 1 \quad \forall t \in \mathbf{R}$  e  $|z| = 1 \Rightarrow \exists ! t \in (-\pi, \pi) : z = \exp(it)$

(i)  $\exp z \exp(-z) = 1$  (ii)  $\exp \bar{z} = \lim_{N \rightarrow +\infty} \sum_{n=0}^N \frac{\bar{z}^n}{n!} = \lim_{N \rightarrow +\infty} \sum_{n=0}^N \frac{z^n}{n!} = \overline{\exp z}$

(iii)  $|\exp(it)|^2 = \exp(it) \overline{\exp(it)} = \exp(it) \exp(-it) = 1$ . Poi,  
 $z = x + iy$ ,  $x^2 + y^2 = 1 \Rightarrow \exists ! t \in [0, \pi] : x = \cos t$  e  $y = \sin t$  se  $y \geq 0$ ,  $x = \cos(-t)$ ,  $y = \sin(-t)$  se  $y < 0$ .

**11. Funzioni circolari ed iperboliche sui complessi**

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbf{C}$$

$$\sinh z := \frac{1}{2}(\exp z - \exp(-z)) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \quad \forall z \in \mathbf{C}$$

$$\cosh z := \frac{1}{2}(\exp z + \exp(-z)) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \quad \forall z \in \mathbf{C}$$

**12.** (i)  $\exp(iz) \equiv \cos z + i \sin z$ ,  $\exp(-iz) \equiv \cos z - i \sin z$

(ii)  $\cos z \equiv \frac{\exp(iz) + \exp(-iz)}{2} \equiv \cosh iz$   $\sin z \equiv \frac{\exp(iz) - \exp(-iz)}{2i} \equiv \frac{\sinh(iz)}{i}$

Da (ii) segue:  $\sin z, \cos z$  sono funzioni  $2\pi$ -periodiche mentre  $\sinh z, \cosh z$  sono  $2\pi i$ -periodiche. Inoltre,  $\sin^2 z + \cos^2 z \equiv 1$ ,  $\cosh^2 z - \sinh^2 z \equiv 1$

**13. Definizione di**  $\arg z, \log z, z \in \mathbf{C}$

Dato  $z \in \mathbf{C}$ ,  $\arg z$  (**argomento di**  $z$ ) é l'unico reale in  $(-\pi, \pi]$  tale che

$$z = |z| \exp(i \arg z)$$

Notiamo che, per periodicitá,  $z = |z| \exp(i(\arg z + 2k\pi)) \quad \forall k \in \mathbf{Z}$ . Scriveremo

$$\text{Arg } z := \{\arg z + 2k\pi, k \in \mathbf{Z}\}$$

Ora, dato  $w \in \mathbf{C}, w \neq 0$

$$\exp z = w \Leftrightarrow \exp(\text{Re } z) \exp(i \text{Im } z) = |w| \exp(i \arg w) \Leftrightarrow$$

$$\exp \text{Re } z = |w| \quad \text{e} \quad \text{Im } z - \arg w \in 2\pi \mathbf{Z} \quad \text{cioé}$$

$$\exp z = w \Leftrightarrow z \in \{\log |w| + i \text{Arg } w\}$$

Porremo  $\text{Log } w := \{\log |w| + i \text{Arg } w\} \quad \forall w \in \mathbf{C}, w \neq 0$

La funzione  $\log w := \log |w| + i \arg w$  si chiama valore principale del logaritmo.

Esempi.  $\text{Log } x = \log x + 2k\pi i, \forall x > 0, \text{Log } x = \log |x| + (2k+1)\pi i, \forall x < 0.$   
 $\log(-1) = \pi i, \log i = \frac{\pi}{2}i, \text{Log}(1-i) = \log \sqrt{2} + (2k - \frac{1}{4})\pi i.$

Esercizi.  $\text{Log}(zw) = \text{Log } z + \text{Log } w$  ove, per  $A, B \subset \mathbf{C}, A + B := \{a + b : a \in A, b \in B\}$ . Infatti  $\text{Arg}(zw) = \text{Arg } z + \text{Arg } w.$

$$\text{Log}(-z) = \text{Log } z + \pi i \quad \forall z \neq 0.$$

Trovare l'errore in  $z^2 = (-z)^2 \Rightarrow \text{Log}(z)^2 = \text{Log}(-z)^2 \Rightarrow$   
 $\text{Log } z + \text{Log } z = \text{Log}(-z) + \text{Log}(-z) \Rightarrow 2\text{Log } z = 2\text{Log}(-z) \Rightarrow \text{Log } z = \text{Log}(-z).$

**14. Potenze in C** Se  $w, z \in \mathbf{C}, w \neq 0$

$$w^z := \exp(z \text{Log } w) = \exp\{z [\log |w| + i(\arg w + 2k\pi)]\} \quad k \in \mathbf{Z}$$

Esempi. Sia  $z = n \in \mathbf{N}$ ;  $w^n = \exp\{n [\log |w| + i(\arg w + 2k\pi)]\} =$   
 $\exp\{n \log |w|\} \exp\{ni(\arg w + 2k\pi)\} = |w|^n [\exp\{i(\arg w + 2k\pi)\}]^n = w \times \dots \times$   
 $w \quad (n \text{ volte}).$

Se  $z = \frac{1}{n}, n \in \mathbf{N}, a^{\frac{1}{n}} = \{|a|^{\frac{1}{n}} \exp i \frac{\arg a + 2k\pi}{n}, k = 0, \dots, n-1\}$  (le  $n$   
 radici complesse di  $a$ ). Se  $z \notin \mathbf{Q}, a^z$  é un insieme infinito. In  
 particolare,  $e^z = \exp z$  se e solo se  $z \in \mathbf{Z}$ .

**APPENDICE**

**A1: Prova di 8.**

Siano  $s_N := \sum_{n=0}^N z_n$ ,  $\sigma_N := \sum_{n=0}^N w_n$

$$p_N := \sum_{n=0}^N \left( \sum_{j+k=n} z_j w_k \right) = z_0 w_0 + (z_0 w_1 + z_1 w_0) + \dots + (z_0 w_N + z_1 w_{N-1} + \dots + z_{N-1} w_1 + z_N w_0)$$

$$= z_0(w_0 + w_1 + \dots + w_N) + z_1(w_0 + \dots + w_{N-1}) + \dots + z_N w_0. \quad \text{Dunque}$$

$$|s_N \sigma_N - p_N| =$$

$$|z_0(w_0 + \dots + w_N) + z_1(w_0 + \dots + w_N) + \dots + z_{N-1}(w_0 + \dots + w_N) + z_N(w_0 + \dots + w_N) -$$

$$[z_0(w_0 + w_1 + \dots + w_N) + z_1(w_0 + \dots + w_{N-1}) + \dots + z_{N-1}(w_0 + w_1) + z_N w_0]| =$$

$$|z_1 w_N + z_2(w_{N-1} + w_N) + \dots + z_{N-1}(w_2 + \dots + w_N) + z_N(w_1 + \dots + w_N)| \leq$$

$$\leq \sum_{j=1}^n \left[ |z_j| \left| \sum_{i=1}^j w_{N-j+i} \right| \right] + \sum_{j=n+1}^N \left[ |z_j| \left| \sum_{i=1}^j w_{N-j+i} \right| \right] \leq$$

$$\leq \left[ \sum_{j=1}^n |z_j| \right] \left[ \sum_{k=N-n+1}^{\infty} |w_k| \right] + \left[ \sum_{j \geq n+1} |z_j| \right] \left[ \sum_{k=1}^{\infty} |w_k| \right] \quad n := \left[ \frac{N}{2} \right]. \quad \text{Da}$$

$$\sum_{k=N-\left[\frac{N}{2}\right]+1}^{\infty} |w_k| \rightarrow_{N \rightarrow +\infty} 0, \quad \sum_{j \geq \left[\frac{N}{2}\right]+1} |z_j| \rightarrow_{N \rightarrow +\infty} 0, \quad \sum_j |z_j| < +\infty, \quad \sum_{k=1}^{\infty} |w_k| < \infty$$

segue  $|s_N \sigma_N - p_N| \rightarrow_{N \rightarrow +\infty} 0$  e quindi  $\lim_N p_N = \lim_N s_N \sigma_N$ .

ESERCIZIO.  $e := \sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n$  É

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \frac{n!}{k! (n-k)!} \frac{1}{n^k} < \sum_{k=0}^n \frac{1}{k!} \quad \text{perché} \quad \frac{n!}{n^k (n-k)!} = \frac{(n-k)! (n-k-1) \dots n}{(n-k)! n \dots n} < 1$$

e quindi  $\limsup_n \left(1 + \frac{1}{n}\right)^n \leq \sum_{n=0}^{\infty} \frac{1}{n!}$

Viceversa,  $n > n_0 \Rightarrow \left(1 + \frac{1}{n}\right)^n > \sum_{k=0}^{n_0} \frac{n!}{k! (n-k)!} \frac{1}{n^k} \Rightarrow \liminf_n \left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^{n_0} \frac{1}{k!}, \quad \forall n_0$

perché  $\frac{n!}{n^k (n-k)!} = \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \left(1 - \frac{k-1}{n}\right) \rightarrow_n 1$ . Quindi

$$\liminf_n \left(1 + \frac{1}{n}\right)^n \geq \sum_{k=0}^{\infty} \frac{1}{k!}$$