Tutorato 9 - ICA
Soluzioni

a) Calcolare i seguenti limiti

1. $\lim_{x\to 1^+} \dfrac{\log(1+\sqrt{x-1})}{\sqrt{x^2-1}} = \dfrac{1}{\sqrt{2}}$

2. $\lim_{x\to 0^+} (1+|\sin x|)^{1/x} = e$

3. $\lim_{x\to\infty} xe^x \sin\left(e^{-x}\sin\dfrac{2}{x}\right) = 2$

4. $\lim_{x\to\frac{\pi}{2}} (1+\cos^2 x)^{\tan^2 x} = e$

5. $\lim_{x\to 0} x\log x = 0$

6. $\lim_{x\to 0} \dfrac{\log\sin x}{\log x} = 1$

7. $\lim_{x\to 0} \dfrac{x^3-3x^2+4x}{x^5-x} = \lim_{x\to 0} \dfrac{x(x^2-3x+4)}{x(x^4-1)} = -4$

8. $\lim_{x\to\infty} \dfrac{x^3-3x}{2x^3-x^2} = \lim_{x\to\infty} \dfrac{x^3(1-\frac{3}{x^2})}{x^3(2-\frac{1}{x})} = \dfrac{1}{2}$

9. $\lim_{x\to\infty} \dfrac{6x^4-x^2}{x-x^3} = \lim_{x\to\infty} \dfrac{x^4(6-\frac{1}{x^2})}{x^3(\frac{1}{x^2}-1)} = -\infty$

10. $\lim_{x\to 0} \dfrac{\sin(\pi+4x)}{x} = \lim_{x\to 0} \dfrac{-4\sin(4x)}{4x} = -4$

11. $\lim_{x\to 0} \dfrac{\cos(\frac{\pi(1-x)}{2})}{x} = \lim_{x\to 0} \dfrac{\cos(\frac{\pi}{2}-\frac{\pi}{2}x)}{x} = \lim_{x\to 0} \dfrac{-\sin(-\frac{\pi}{2}x)}{x} = \lim_{x\to 0} \dfrac{\frac{\pi}{2}\sin(\frac{\pi}{2}x)}{\frac{\pi}{2}x} =$
$= \dfrac{\pi}{2}$

12. $\lim_{x\to 0} \dfrac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{\sin x} =$

$$= \lim_{x\to 0} \frac{\sqrt{1+\tan x}-\sqrt{1-\tan x}}{\sin x} \cdot \frac{\sqrt{1+\tan x}+\sqrt{1-\tan x}}{\sqrt{1+\tan x}+\sqrt{1-\tan x}} =$$

$$= \lim_{x\to 0} \frac{1+\tan x-1+\tan x}{\sin x(\sqrt{1+\tan x}+\sqrt{1-\tan x})} =$$

$$= \lim_{x\to 0} \frac{2}{\cos x(\sqrt{1+\tan x}+\sqrt{1-\tan x})} = 1$$

13. $\lim_{x\to\infty} (\sqrt{x}-1+\cos x) = \infty$

perché $\qquad \sqrt{x} - 1 + \cos x \geqslant \sqrt{x} - 2 \to \infty \qquad$ per $x \to \infty$

14. $\displaystyle \lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

15. $\displaystyle \lim_{x\to\frac{\pi}{2}} (1+\cos^2 x)^{\tan^2 x} = \lim_{x\to\frac{\pi}{2}} (1+\cos^2 x)^{\frac{1}{\cos^2 x}\sin^2 x} = \lim_{x\to\frac{\pi}{2}} e^{\sin^2 x} = e$

16. $\displaystyle \lim_{x\to 0} (1+x)^{\tan x} = \lim_{x\to 0} (1+x)^{\frac{1}{x}\cdot x\tan x} = \lim_{x\to 0} e^{x\tan x} = 1$

17. $\displaystyle \lim_{x\to 0} \frac{\log\cos x}{x^2} = \lim_{x\to 0} \frac{\log(1+\cos x - 1)}{\cos x - 1}\cdot\frac{\cos x - 1}{x^2} = -\frac{1}{2}$

18. $\displaystyle \lim_{x\to\infty} \frac{\log(3+\sin x)}{x^3} = 0$

perché $\displaystyle \qquad \frac{\log(2)}{x^3} \leqslant \frac{\log(3+\sin x)}{x^3} \leqslant \frac{\log(4)}{x^3}$

19. $\displaystyle \lim_{x\to-\infty} \frac{3^x - 3^{-x}}{3^x + 3^{-x}} = \lim_{x\to-\infty} \frac{3^x - \frac{1}{3^x}}{3^x + \frac{1}{3^x}} = \lim_{x\to-\infty} \frac{\frac{1}{3^x}(3^{2x}-1)}{\frac{1}{3^x}(3^{2x}+1)} =$

$$= \lim_{x\to-\infty} \frac{3^{2x}-1}{3^{2x}+1} = -1$$

20. $\displaystyle \lim_{x\to 5} \frac{x-5}{\sqrt{x}-\sqrt{5}} = \lim_{x\to 5} \frac{(\sqrt{x}-\sqrt{5})(\sqrt{x}+\sqrt{5})}{\sqrt{x}-\sqrt{5}} = \lim_{x\to 5} \sqrt{x}+\sqrt{5} = 2\sqrt{5}$

b) Calcolare i seguenti limiti usando lo sviluppo di Taylor:

1. $\displaystyle \lim_{x\to 0} \frac{1 - e^{-x^2} + x^3\sin(1/x)}{x^2} = 1$

2. $\displaystyle \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^3(e^x - \cos x)} = \lim_{x\to 0} \frac{x^2 - (x - \frac{x^3}{6} + o(x^4))^2}{x^3(1 + x + o(x) - 1 + o(x))} = \lim_{x\to 0} \frac{x^2 - (x^2 - \frac{x^4}{3} + o(x^5))}{x^3(x + o(x))} =$

$$= \lim_{x\to 0} \frac{\frac{x^4}{3} + o(x^5)}{x^4(1+o(1))} = \lim_{x\to 0} \frac{x^4(\frac{1}{3} + o(x))}{x^4(1+o(1))} = \frac{1}{3}$$

3. $\displaystyle \lim_{x\to 0} \frac{\log(1+x)\arctan x - x\sin x}{\arctan x - 1 - \log(1+x) + \cos x} = \frac{3}{4}$

4. $\displaystyle \lim_{x\to 0^+} \frac{\log(1-\cos 2x)}{\log\tan 2x} = 2$

5. $\displaystyle \lim_{x\to 0^+} \frac{\sqrt[4]{1+\sin^2 x} - 1}{\log\left[1 + \sqrt{1 - e^{-x^2}}\right]\left[(1+\sin x)^{-1/x} - e^{-1}\right]} =$

$$= \lim_{x\to 0^+} \frac{\sqrt[4]{1 + (x + o(x^2))^2} - 1}{\log\left[1 + \sqrt{1 - 1 + x^2 + o(x^2)}\right]\left[(1 + x + o(x^2))^{-1/x} - e^{-1}\right]} =$$

$$= \lim_{x\to 0^+} \frac{\sqrt[4]{1+x^2+o(x^3)}-1}{\log\left[1+\sqrt{x^2(1+o(1))}\right]\left[e^{\log(1+x+o(x^2))^{-1/x}}-e^{-1}\right]} =$$

$$= \lim_{x\to 0^+} \frac{1+\frac{1}{4}(x^2+o(x^3)+o(x^2))-1}{\log\left[1+x\sqrt{1+o(1)}\right]\left[e^{-\frac{\log(1+x+o(x^2))}{x}}-e^{-1}\right]} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{4}(x^2+o(x^2))}{\log\left[1+x(1+o(1))\right]\left[e^{-\frac{x+o(x^2)-\frac{1}{2}(x+o(x^2))^2+o(x^2)}{x}}-e^{-1}\right]} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{4}x^2+o(x^2)}{\log\left[1+x+o(x)\right]\left[e^{-\frac{x-\frac{1}{2}(x^2+o(x^3))+o(x^2)}{x}}-e^{-1}\right]} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{4}x^2+o(x^2)}{\left[x+o(x)\right]\left[e^{-\frac{x-\frac{1}{2}x^2+o(x^2)}{x}}-e^{-1}\right]} = \lim_{x\to 0^+} \frac{\frac{1}{4}x^2+o(x^2)}{x\left[1+o(1)\right]\left[e^{-1+\frac{1}{2}x+o(x)}-e^{-1}\right]} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{4}x^2+o(x^2)}{x\left[1+o(1)\right]e^{-1}\left[e^{\frac{1}{2}x+o(x)}-1\right]} = \lim_{x\to 0^+} \frac{\frac{1}{4}x^2+o(x^2)}{\frac{x}{e}\left[1+o(1)\right]\left[1+\frac{1}{2}x+o(x)-1\right]} =$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{4}x^2+o(x^2)}{\frac{x}{e}\left[1+o(1)\right]\left[\frac{1}{2}x+o(x)\right]} = \lim_{x\to 0^+} \frac{x^2(\frac{1}{4}+o(1))}{x^2\left[1+o(1)\right]\left[\frac{1}{2e}+o(1)\right]} = \frac{e}{2}$$

6. $\displaystyle \lim_{x\to 0} \frac{(\arcsin x)^2+\log(1-\sin^2 x)}{\cosh x^2-1} = \frac{1}{3}$