**Università degli Studi di Roma TreCorso di laurea in Matematica**
**Tutorato di ST1 - A.A. 2007/2008**
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Tutorato n.1 del 29/02/2008 -Soluzioni

**Esercizio 1**

1. $P(X > n) = P(\text{croce nei primi n lanci}) = (1-p)^n$

2. $$F_X(x) = P(X \le x) = P(X \le [x]) = 1 - P(X > [x]) = \begin{cases} 1-(1-p)^{[x]} & x > 0 \\ 0 & x \le 0 \end{cases}$$

**Esercizio 2**

Osserviamo che $F(x)$ è continua.

1. $F\left(\frac{3}{2}\right) - F\left(\frac{1}{2}\right) = \frac{1}{2}$
2. $P(Y \le X) = P(X^2 - X \le 0) = P(X(X-1) \le 0) = P(0 \le X \le 1) = F(1) - F(0) = \frac{1}{2}$
3. $P\left(X + Y \le \frac{3}{4}\right) = P\left(-\frac{3}{2} \le X \le \frac{1}{2}\right) = F\left(\frac{1}{2}\right) = \frac{1}{4}$
4. $$P(Z \le z) = P(\sqrt{X} \le z) = P(X \le z^2) = \begin{cases} 0 & z < 0 \\ \frac{1}{2}z^2 & 0 \le z \le \sqrt{2} \\ 1 & z > \sqrt{2} \end{cases}$$

**Esercizio 3**

$$P(Z = z) = P(X+Y = z) = \sum_{k=0}^{n} P(X = k, Y = z-k) = \sum_{k=0}^{n} P(X = k)P(Y = z-k) =$$
$$\sum_{k=0}^{z} \frac{e^{-\lambda}\lambda^k}{k!}\frac{e^{-\lambda}\lambda^{z-k}}{z-k!} = \frac{e^{-\lambda}e^{-\mu}}{z!}\sum_{k=0}^{z}\begin{pmatrix} z \\ k \end{pmatrix}\lambda^k\mu^{z-k} = \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^z}{z!}$$

**Esercizio 4**

1. $$\int_0^1 C\frac{1}{\sqrt{x(1-x)}}dx = \int_0^1 C\frac{1}{\sqrt{x-x^2}}dx = \int_0^1 C\frac{2}{\sqrt{4x-4x^2}}dx =$$
$$= C\int_0^1 \frac{2}{\sqrt{1-4x^2+4x-1}}dx = C\int_0^1 \frac{2}{\sqrt{1-(2x-1)^2}}dx = C\int_0^1 \frac{d}{dx}arcsin(2x-1)dx = C\pi$$
   $f_X(x)$ è funzione di densità se e solo se $C = \frac{1}{\pi}$

2. $$\int_{-\infty}^{\infty} Ce^{-x-e^{-x}}dx = lim_{k\to\infty}\int_{-k}^{k} Ce^{-x-e^{-x}}dx = Clim_{k\to\infty}\left[-e^{-e^{-x}}\right]_{-k}^{k} = 1$$

   Quindi basta prendere $C = 1$

**Esercizio 5**

1. $X$ ha funzione di densità $f_X(x) = \mathbf{1}_{[0,1]}(x)$.
   Analogamente $Y$ ha funzione di densità $f_Y(y) = \mathbf{1}_{[0,1]}(y)$.
   Calcoliamola funzione di distribuzione di $U$:

$$P(U \le u) = P(min\{X, Y\} \le u) = 1 - P(min\{X, Y\} \ge u) =$$

$$= 1 - P(X \ge u, Y \ge u) = 1 - [P(X \ge u)]^2 = 1 - \left(\int_u^{\infty} \mathbf{1}_{[0,1]}(x)dx\right)^2 =$$

$$= \begin{cases} 1 - [1-u]^2 & se \quad u \in [0,1] \\ 1 & se \quad u \ge 1 \\ 0 & se \quad u \le 0 \end{cases}$$

$$f_U(u) = \frac{d}{du}P(U \le u) = (2-2u)\mathbf{1}_{[0,1]}(u)$$

$$\mathbb{E}(U) = \int_0^1 u(2-2u)du = \frac{1}{3}$$

   Calcoliamola funzione di distribuzione di $V$:

$$P(V \le v) = P(max\{X, Y\} \le v) = P(X \le v, Y \le v) = [P(X \le v)]^2 = \left(\int_{-\infty}^{u} \mathbf{1}_{[0,1]}(x)dx\right)^2 =$$

$$= \begin{cases} v^2 & se \quad v \in [0,1] \\ 1 & se \quad v \ge 1 \\ 0 & se \quad v \le 0 \end{cases}$$

$$f_V(v) = \frac{d}{dv}P(V \le v) = 2v\mathbf{1}_{[0,1]}(v)$$

$$\mathbb{E}(V) = \int_0^1 2v^2 dv = \frac{2}{3}$$

$$Cov(U, V) = \mathbb{E}(UV) - \mathbb{E}(U)\mathbb{E}(V)$$

Osserviamo che $UV = XY$,
quindi $\mathbb{E}(UV) = \mathbb{E}(XY) = E(X)E(Y) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.
Dunque $Cov(U, V) = \frac{1}{4} - \frac{1}{3}\frac{2}{3} = \frac{1}{36}$.

**Esercizio 6** *1.*

$$f_{X,Y}(X,Y) = 2\mathbf{1}_{(0,y)}(x)\mathbf{1}_{(0,1)}(y) = 2\mathbf{1}_{(0,1)}(x)\mathbf{1}_{(x,1)}(y)$$

$$f_X(x) = \int_{-\infty}^{+\infty} 2\mathbf{1}_{(0,1)}(x)\mathbf{1}_{(x,1)}(y) = 2\mathbf{1}_{(0,1)}(x)\int_x^1 dy = (2-2x)\mathbf{1}_{(0,1)}(x)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} 2\mathbf{1}_{(0,1)}(y)\mathbf{1}_{(0,y)}(x) = 2\mathbf{1}_{(0,1)}(y)\int_0^y dx = 2y\mathbf{1}_{(0,1)}(y)$$

$$\mathbf{E}(X) = \int_0^1 x(2-2x)dx = \frac{1}{3}$$

$$\mathbf{E}(Y) = \int_0^1 2y^2 dy = \frac{2}{3}$$

$$Cov(X,Y) = \int_0^1 \left(\int_0^y 2\left(x-\frac{1}{3}\right)\left(y-\frac{2}{3}\right)dx\right)dy = \frac{1}{36}$$

*2.*

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2\mathbf{1}_{(0,1)}(x)\mathbf{1}_{(x,1)}(y)}{(2-2x)\mathbf{1}_{(0,1)}(x)} = \frac{1}{1-x}\mathbf{1}_{(x,1)}(y)$$

**Esercizio 7**

1. Per simmetria $f_X(x) = f_Y(y)$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)dy = \mathbf{1}_{[0,+\infty]}(x)\left[\int_x^{+\infty} e^{-y}\left(1-e^{-x}\right)dy + \int_0^x e^{-x}(1-e^{-y})dy\right] =$$

$$xe^{-x}\mathbf{1}_{[0,+\infty]}(x)$$

2. Per simmetria $\mathbb{E}(X) = \mathbb{E}(Y)$

$$\mathbb{E}(X) = \int_0^{+\infty} x{\cdot}x{\cdot}e^{-x}dx = \left[-x^2e^{-x}\right]_0^{+\infty} + 2\int_0^{+\infty} xe^{-x}dx = \left[-2xe^{-x}\right]_0^{+\infty} + 2\int_0^{+\infty} e^{-x}dx =$$

$$\left[-2e^{-x}\right]_0^{+\infty} = 2$$

3. Per simmetria $Var(X) = Var(Y)$

$$\mathbb{E}(X^2) = \int_0^{+\infty} x^3e^{-x}dx = \left[-x^3e^{-x}\right]_0^{+\infty} + 3\int_0^{+\infty} x^2e^{-x}dx =$$

$$3 \cdot 2 = 6$$

$$Var(X) = 6 - 2^2 = 2$$

4.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{VarXVar(Y)}}$$

$$Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

$$\mathbb{E}(XY) = 2\left[\int_0^{+\infty} ye^{-y}\left(\int_0^y x(1-e^{-x})dx\right)dy\right] = 5$$

$$\rho(X,Y) = \frac{5-4}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

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