Università degli studi Roma Tre - Corso di Laurea in Matematica
Tutorato di ST1 - A.A. 2005/2006
Docente: Prof.ssa E. Scoppola - Tutore: Dott. Nazareno Maroni

Soluzioni del tutorato n.4 del 30/3/2006

Esercizio 1. Sappiamo che $\mathbb{E}(X_i) = \frac{\alpha}{\alpha+\beta}$ e $\mathbb{E}(X^2) = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$, quindi risolviamo il sistema:

$$\begin{cases} m_1' = \frac{\alpha}{\alpha+\beta} \\ m_2' = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \end{cases}$$

Risolvendo si ha: $\hat{\alpha} = m_1' \frac{m_2'-m_1'}{m_1'^2-m_2'}$ e $\hat{\beta} = (1-m_1')\frac{m_2'-m_1'}{m_1'^2-m_2'}$.

Esercizio 2.

$$L(\lambda) = \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n e^{-n\lambda \sum_i x_i} = \lambda^n e^{-n\lambda \overline{X}}$$

$$\Rightarrow \log L(\lambda) = n \log \lambda - n\lambda \overline{X}$$

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{n}{\lambda} - n\overline{X} = 0 \quad \Leftrightarrow \quad n(1-\overline{X}\lambda) = 0 \quad \Leftrightarrow \quad \hat{\lambda} = \frac{1}{\overline{X}}$$

$$\Rightarrow \quad \hat{\Lambda} = \frac{n}{\sum_i X_i}$$

Usando il teorema delle riparametrizzazioni: $\hat{\Psi} = \frac{\sum_i X_i}{n}$. Sappiamo che $\mathbb{E}(X_i) = \frac{1}{\lambda}$ quindi $\frac{1}{\lambda} = \overline{X}$ cioé $\hat{\lambda} = \frac{1}{\overline{X}}$.

Esercizio 3.

$$L(\lambda) = \prod_{i=1}^n \frac{\lambda^{x_i}}{x_i!} e^{-\lambda} = \frac{\lambda^{\sum_i x_i}}{\prod_i x_i!} e^{-n\lambda}$$

$$\Rightarrow \log L(\lambda) = -\log\left(\prod_{i=1}^n x_i!\right) + \sum_{i=1}^n x_i \ \log \lambda - n\lambda$$

$$\frac{d}{d\lambda} \log L(\lambda) = \frac{\sum_i x_i}{\lambda} - n = 0 \quad \Leftrightarrow \quad \sum_{i=1}^n x_i - n\lambda = 0 \quad \Leftrightarrow \quad \hat{\lambda} = \frac{\sum_i x_i}{n}$$

$$\Rightarrow \quad \hat{\Lambda} = M_1'$$

Poiché $\mathbb{E}(X_i) = \lambda$ si ha con il metodo dei momenti: $\hat{\lambda} = m_1'$.

Esercizio 4.

$$L(p) = \prod_{i=1}^{n} p(1-p)^{x_i} = p^n(1-p)^{\sum_i x_i}$$

$$\Rightarrow \quad \log L(p) = n \log p + \sum_{i=1}^{n} x_i \ \log(1-p)$$

$$\frac{d}{dp} \log L(p) = \frac{n}{p} - \frac{\sum_i x_i}{1-p} = 0 \quad \Leftrightarrow \quad \hat{p} = \frac{n}{\sum_i x_i + n}$$

$$\Rightarrow \quad \hat{P} = \frac{n}{\sum_i X_i + n}$$

Poiché $\mathbb{E}(X_i) = \frac{1-p}{p}$ si ha con il metodo dei momenti: $m_1' = \frac{1-p}{p}$ e quindi $\hat{p} = \frac{n}{\sum_i x_i + n}$.

Esercizio 5.

$$L(\beta, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - \beta x_i)^2} = \frac{1}{(2\pi)^{n/2}(\sigma^2)^{n/2}} e^{-\frac{1}{2\sigma^2}\sum_i (y_i - \beta x_i)^2}$$

$$\Rightarrow \quad \log L(\beta, \sigma^2) = -\frac{n}{2} \log 2\pi - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta x_i)^2$$

$$\frac{d}{d\beta} \log L(\beta, \sigma^2) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} [2(y_i - \beta x_i)(-x_i)] = 0 \quad \Leftrightarrow \quad \hat{\beta} = \frac{\sum_i x_i y_i}{\sum_i x_i^2}$$

$$\frac{d}{d\sigma^2} \log L(\beta, \sigma^2) = -\frac{n}{2}\frac{1}{\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} (x_i y_i - \hat{\beta} x_i)^2 = 0 \quad \Leftrightarrow \quad \hat{\sigma}^2 = \frac{\sum_i (y_i - \hat{\beta} x_i)^2}{n}$$

$\mathbb{E}(\hat{\beta}) = \beta$ e $Var(\hat{\beta}) = \frac{\sigma^2}{\sum_i x_i^2}$ e poiché è somma di normali $\hat{\beta} \sim N(\beta, \frac{\sigma^2}{\sum_i x_i^2})$.

Esercizio 6.

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i e^{-x/\theta} = \frac{1}{\theta^{2n}} \prod_{i=1}^{n} x_i e^{-\frac{1}{\theta}\sum_i x_i}$$

$$\Rightarrow \quad \log L(\theta) = -2n \log \theta + log \prod_{i=1}^{n} x_i - \frac{1}{\theta} \sum_{i=1}^{n} x_i$$

$$\frac{d}{d\theta} \log L(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i = 0 \quad \Leftrightarrow \quad \hat{\theta} = \frac{\sum_i x_i}{2n}$$

Poiché è $\mathbb{E}(X_i) = 2\theta$ si ha con il metodo dei momenti $m_1' = 2\theta$ e quindi $\hat{\theta} = \frac{\sum_i x_i}{2n}$.