Università degli studi Roma Tre - Corso di Laurea in Matematica
Soluzioni del tutorato di ST1 - A.A. 2005/2006
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Esercizio 1. Sappiamo che $U = \frac{n-1}{\sigma^2}S^2 \sim \chi^2_{n-1}$, quindi calcoliamo

$$\mathbb{E}(\sqrt{U}) = \int_0^{+\infty} u^{\frac{1}{2}} \frac{u^{\frac{n-1}{2}-1}e^{-\frac{1}{2}u}}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}du = \int_0^{+\infty} \frac{u^{\frac{n}{2}-1}e^{-\frac{1}{2}u}}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}du =$$
$$= \int_0^{+\infty} \frac{\sqrt{2}}{\Gamma(\frac{n-1}{2})} \frac{u^{\frac{n}{2}-1}e^{-\frac{1}{2}u}}{2^{\frac{n}{2}}}du = \frac{\sqrt{2}\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}$$

Però è anche $\mathbb{E}(\sqrt{U}) = \mathbb{E}\left(\sqrt{\frac{n-1}{\sigma^2}S^2}\right) = \frac{\sqrt{n-1}}{\sigma}\mathbb{E}(S)$, quindi $\mathbb{E}(S) = \frac{\sigma}{\sqrt{n-1}}\mathbb{E}(\sqrt{U}) = \frac{\sqrt{2}\sigma}{\sqrt{n-1}}\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}$.

$$Var(\sqrt{U}) = \mathbb{E}(U) - (\mathbb{E}(\sqrt{U}))^2 = (n-1) - 2\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right)^2 \quad \text{e quindi}$$
$$Var(\sqrt{U}) = Var\left(\frac{\sqrt{n-1}}{\sigma}S\right) = \frac{n-1}{\sigma^2}Var(S) \quad \Rightarrow$$
$$Var(S) = \frac{\sigma^2}{n-1}\left[(n-1) - 2\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right)^2\right] = \sigma^2\left[1 - \frac{2}{n-1}\left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}\right)^2\right]$$

Esercizio 2.

(a) $F_Y(y) = \mathbb{P}(\frac{1}{X} \leqslant y) = \mathbb{P}(X \geqslant \frac{1}{y}) = 1 - F_X(\frac{1}{y}) \quad \Rightarrow$

$$f_Y(y) = f_X(1/y)\frac{1}{y^2} = \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{m}{n}\right)^{\frac{m}{2}} \frac{(\frac{1}{y})^{\frac{m-2}{2}}\frac{1}{y^2}}{(1+\frac{m}{n}\frac{1}{y})^{\frac{m+n}{2}}} =$$
$$= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{m}{n}\right)^{\frac{m}{2}} \frac{(\frac{1}{y})^{\frac{m+2}{2}}}{(\frac{m}{n}\frac{1}{y})^{\frac{m+n}{2}}(\frac{n}{m}y+1)^{\frac{m+n}{2}}} =$$
$$= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\left(\frac{n}{m}\right)^{\frac{n}{2}} \frac{y^{\frac{n-2}{2}}}{(1+\frac{n}{m}y)^{\frac{m+n}{2}}} \quad \Rightarrow \quad Y \sim F(n,m)$$

(b) $F_W(w) = \mathbb{P}(W \leqslant w) = \mathbb{P}(\frac{m}{n}X \leqslant (1+\frac{m}{n}X)w) = \mathbb{P}(\frac{m}{n}X(1-w) \leqslant w) =$

$$\mathbb{P}(X \leqslant \tfrac{n}{m}\tfrac{w}{1-w}) = F_X(\tfrac{n}{m}\tfrac{w}{1-w}) \quad \Rightarrow$$

$$f_W(w) = f_X(\frac{n}{m}\frac{w}{1-w})\frac{n}{m}\frac{1}{(1-w)^2} =$$

$$= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\frac{n}{m}\left(\frac{m}{n}\right)^{\frac{m}{2}} \frac{\left(\frac{n}{m}\frac{w}{1-w}\right)^{\frac{m-2}{2}}\frac{1}{(1-w)^2}}{(1+\frac{m}{n}\frac{n}{m}\frac{w}{1-w})^{\frac{m+n}{2}}} =$$

$$= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\frac{n}{m}\left(\frac{m}{n}\right)^{\frac{m}{2}}\left(\frac{n}{m}\right)^{\frac{m-2}{2}} \frac{\left(\frac{w}{1-w}\right)^{\frac{m-2}{2}}\frac{1}{(1-w)^2}}{\left(\frac{1}{1-w}\right)^{\frac{m+n}{2}}} =$$

$$= \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m-2}{2}}(1-w)^{\frac{m+n}{2}-\frac{m-2}{2}-2} =$$

$$= \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m}{2}-1}(1-w)^{\frac{n}{2}-1} \quad \Rightarrow \quad W \sim Beta(\frac{m}{2},\frac{n}{2})$$

(c) Poiché $W = \frac{mX/n}{1+mX/n}$, si vede che $\frac{W}{1-W} = m\frac{X}{n}$ calcoliamo

$$\mathbb{E}\left(m\frac{X}{n}\right) = \mathbb{E}\left(\frac{W}{1-W}\right) = \int_0^1 \frac{w}{1-w}\frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m}{2}-1}(1-w)^{\frac{n}{2}-1}dw =$$

$$= \int_0^1 \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m}{2}}(1-w)^{\frac{n}{2}-2}dw =$$

$$= \int_0^1 \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m+2}{2}-1}(1-w)^{\frac{n-2}{2}-1}dw =$$

$$= \frac{\Gamma(\frac{m}{2}+\frac{n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}\frac{\Gamma(\frac{m+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(\frac{m+2}{2}+\frac{n-2}{2})} = \frac{\Gamma(\frac{m+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}$$

$$\Rightarrow \quad \mathbb{E}(X) = \frac{n}{m}\frac{\Gamma(\frac{m+2}{2})\Gamma(\frac{n-2}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} = \frac{n}{n-2} \quad \text{per la proprietà della } \Gamma$$

$$\mathbb{E}\left(\frac{w^2}{(1-w)^2}\right) = \int_0^1 \frac{w^2}{(1-w)^2}\frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m}{2}-1}(1-w)^{\frac{n}{2}-1}dw =$$

$$= \int_0^1 \frac{\Gamma(\frac{m+n}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} w^{\frac{m+4}{2}-1}(1-w)^{\frac{n-4}{2}-1}dw =$$

$$= \frac{\Gamma(\frac{m+4}{2})\Gamma(\frac{n-4}{2})}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})} = \frac{m(m+2)}{(n-2)(n-4)}$$

$$\mathbb{E}\left(\frac{w^2}{(1-w)^2}\right) = \mathbb{E}\left(\frac{m^2}{n^2}X^2\right) = \frac{m^2}{n^2}\mathbb{E}(X^2) \quad \Rightarrow$$

$$\mathbb{E}(X^2) = \frac{n^2}{m}\frac{m+2}{(n-2)(n-4)} \quad \Rightarrow$$

$$
\begin{aligned}
Var(X) &= \frac{n^2}{m}\frac{m+2}{(n-2)(n-4)} - \frac{n^2}{(n-2)^2} = \\
&= \frac{n^2}{n-2}\frac{(n-2)(m+2)-m(n-4)}{m(n-2)(n-4)} = \\
&= \frac{n^2}{n-2}\frac{nm+2n-2m-4-nm+4m}{m(n-2)(n-4)} = \frac{2n^2(n+m-2)}{m(n-2)^2(n-4)}
\end{aligned}
$$

Esercizio 3.

$$X = \frac{Z}{\sqrt{U/k}} \quad \text{dove} \quad Z \sim N(0,1) \quad \text{e} \quad U \sim \chi^2 - k$$

$$\mathbb{E}\left(\frac{Z}{\sqrt{U/k}}\right) = \mathbb{E}(Z)\mathbb{E}\left(\frac{1}{\sqrt{U/k}}\right) = 0$$

$$\mathbb{E}\left(\frac{1}{\sqrt{U/k}}\right) = \int_0^{+\infty} \frac{\sqrt{k}}{\sqrt{U}}\frac{1}{\sqrt{k\pi}}\frac{u^{\frac{k}{2}-1}e^{-\frac{1}{2}u}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})}du = \int_0^{+\infty} \frac{u^{\frac{k-3}{2}}e^{-\frac{1}{2}u}}{\sqrt{\pi}\,2^{\frac{k}{2}}\Gamma(\frac{k}{2})}du$$

$$\Rightarrow \quad \frac{k-3}{2} > -1 \quad \Leftrightarrow \quad k > 1$$

$$Var\left(\frac{Z}{\sqrt{U/k}}\right) = \mathbb{E}\left(\frac{Z^2}{U/k}\right) = \mathbb{E}(Z^2)\mathbb{E}\left(\frac{1}{U/k}\right) = \frac{k}{k-2}$$

$$
\begin{aligned}
\mathbb{E}\left(\frac{1}{\sqrt{U/k}}\right) &= \int_0^{+\infty} \frac{k}{u}\frac{u^{\frac{k}{2}-1}e^{-\frac{1}{2}u}}{\sqrt{k\pi}\,2^{\frac{k}{2}}\Gamma(\frac{k}{2})}du = \int_0^{+\infty} \frac{k\,u^{\frac{k}{2}-2}e^{-\frac{1}{2}u}}{\sqrt{k\pi}\,2^{\frac{k}{2}}\Gamma(\frac{k}{2})}du = \\
&= \int_0^{+\infty} \frac{k}{2\,\Gamma(\frac{k}{2})}\frac{u^{\frac{k-2}{2}-1}e^{-\frac{1}{2}u}}{\sqrt{k\pi}\,2^{\frac{k-2}{2}}}du = \frac{k}{2}\frac{\Gamma(\frac{k-2}{2})}{\Gamma(\frac{k}{2})} = \frac{k}{k-2} \quad \Leftrightarrow
\end{aligned}
$$

$$\frac{k-2}{2} - 1 > -1 \quad \Leftrightarrow \quad k > 2$$

Esercizio 4.

$$
\begin{aligned}
\mathbb{E}(X) &= \int_0^1 x\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+1-1}(1-x)^{\beta-1}dx = \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} = \frac{\alpha}{\alpha+\beta} \\
\mathbb{E}(X^2) &= \int_0^1 x^2\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}dx = \int_0^1 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha+2-1}(1-x)^{\beta-1}dx = \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\beta+2)} = \frac{\alpha(\alpha+1)}{(\alpha+\beta+1)(\alpha+\beta)}
\end{aligned}
$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$