Università degli studi Roma Tre - Corso di Laurea in Matematica
Tutorato di ST1 - A.A. 2006/2007
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Soluzioni del tutorato n.2 del 13/3/2007

Esercizio 1.

$$\mathbb{E}(X_1) = \int_{-\infty}^{+\infty} x_1 x_1^{-2} \mathbb{1}_{(1,+\infty)}(x_1)\,\mathrm{d}x_1 = \int_1^{+\infty} x_1^{-1}\,\mathrm{d}x_1 = +\infty$$

Quindi $\mathbb{E}(X_1) \nexists$ finito. Per calcolare $\mathbb{E}(Y)$ ci serve la sua distribuzione. Ovviamente è $1 < Y < +\infty$.

$$\begin{aligned}
F_Y(y) = \mathbb{P}(Y \leqslant y) = \mathbb{P}\big(\min_i X_i \leqslant y\big) &= 1 - \mathbb{P}(\min_i X_i > y) = \\
&= 1 - \mathbb{P}(X_1 > y, X_2 > y, \dots, X_n > y) = 1 - \prod_{i=1}^n \mathbb{P}(X_i > y) = \\
&= 1 - \Big(\mathbb{P}(X > y)\Big)^n = 1 - \left(\int_y^{+\infty} \frac{1}{x^2}\,\mathrm{d}x\right)^n = 1 - \frac{1}{y^n} \\
f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y} F_Y(y) = \frac{n}{y^{n+1}} \\
\mathbb{E}(Y) &= \int_1^{+\infty} y \frac{n}{y^{n+1}}\,\mathrm{d}y = \int_1^{+\infty} \frac{n}{y^n}\,\mathrm{d}y = \frac{n}{n-1}
\end{aligned}$$

Esercizio 2. Calcoliamo la f.g.m. di una $X \sim \mathit{Gamma}(r, \lambda)$:

$$\begin{aligned}
\mathbb{E}(e^{tX}) &= \int_0^{+\infty} e^{tx} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}\,\mathrm{d}x = \\
&= \lambda^r \int_0^{+\infty} \frac{1}{\Gamma(r)} x^{r-1} e^{-(\lambda - t)x} \frac{(\lambda - t)^r}{(\lambda - t)^r}\,\mathrm{d}x = \Big(\frac{\lambda}{\lambda - t}\Big)^r
\end{aligned}$$

(a)

$$\begin{aligned}
\mathbb{E}(e^{tY}) = \mathbb{E}(e^{t(X_1 + \dots + X_n)}) = \mathbb{E}(e^{tX_1} \cdot \ldots \cdot e^{tX_n}) &= \mathbb{E}(e^{tX_1}) \cdot \ldots \cdot \mathbb{E}(e^{tX_n}) = \\
= \prod_{i=1}^n \Big(\frac{\lambda}{\lambda - t}\Big)^r &= \Big(\frac{\lambda}{\lambda - t}\Big)^{rn} \Rightarrow Y \sim \mathit{Gamma}(rn, \lambda)
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbb{E}(e^{tY}) = \mathbb{E}(e^{t(X_1 + \dots + X_n)}) = \mathbb{E}(e^{tX_1} \cdot \ldots \cdot e^{tX_n}) &= \mathbb{E}(e^{tX_1}) \cdot \ldots \cdot \mathbb{E}(e^{tX_n}) = \\
= \prod_{i=1}^n \Big(\frac{\lambda}{\lambda - t}\Big)^{r_i} &= \Big(\frac{\lambda}{\lambda - t}\Big)^{\sum r_i} \Rightarrow Y \sim \mathit{Gamma}\Big(\sum_{i=1}^n r_i, \lambda\Big)
\end{aligned}$$

Esercizio 3. Abbiamo che $Z^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$, la distribuzione congiunta di X_1 e X_2 è una Normale bivariata così come la congiunta di Y_1 e Y_2

$$\begin{aligned}
F_{X_2-X_1}(x) &= \mathbb{P}(X_2 - X_1 \leqslant x) = \mathbb{E}(\mathbb{1}_{\{x_2-x_1\leqslant x\}}(x_1,x_2)) = \\
&= \iint\limits_{x_2-x_1\leqslant x} \frac{1}{2\pi} e^{-\frac{x_1^2+x_2^2}{2}}\,\mathrm{d}x_1\mathrm{d}x_2 = \int_{-\infty}^{+\infty}\mathrm{d}x_1\int_{-\infty}^{x_1+x}\mathrm{d}x_2\,\frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}} \\
f_{X_2-X_1}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}F_{X_2-X_1}(x) = \int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{-\frac{x_1^2+(x_1+x)^2}{2}}\,\mathrm{d}x_1 = \\
&= \int_{-\infty}^{+\infty}\frac{1}{2\pi}e^{-\frac{2x_1^2+2xx_1}{2}}e^{-\frac{x^2}{2}}\,\mathrm{d}x_1 = \int_{-\infty}^{+\infty}\frac{1}{2\pi}\frac{\sqrt{2}}{\sqrt{2}}e^{-2\frac{(x_1-\frac{-x}{2})^2}{2}+\frac{x^2}{4}}e^{-\frac{x^2}{2}}\,\mathrm{d}x_1 = \\
&= \frac{1}{\sqrt{2\pi}\sqrt{2}}e^{-\frac{x^2}{4}} \sim N(0,2) \\
f_{Y_2-Y_1}(y) &\sim N(0,2)
\end{aligned}$$

$Z_1 = X_2 - X_1$ e $Z_2 = Y_2 - Y_1$ sono indipendenti.

$$\begin{aligned}
f_{Z_1^2}(z) &= 2\frac{1}{\sqrt{2\pi}\sqrt{2}}e^{-\frac{z}{4}}\frac{1}{2\sqrt{z}} = \left(\frac{1}{4}\right)^{\frac{1}{2}}\frac{1}{\Gamma\left(\frac{1}{2}\right)}z^{\frac{1}{2}-1}e^{-\frac{z}{4}} \Rightarrow Z_1^2 \sim Gamma\Big(\frac{1}{2},\frac{1}{4}\Big) \\
f_{Z_2^2} &\sim Gamma\Big(\frac{1}{2},\frac{1}{4}\Big)
\end{aligned}$$

Sappiamo che la f.g.m. di una $Gamma\big(\frac{1}{2},\frac{1}{4}\big)$ è $\left(\frac{\frac{1}{4}}{\frac{1}{4}-t}\right)^{\frac{1}{2}}$.

$$\begin{aligned}
\mathbb{E}(e^{tZ^2}) &= \mathbb{E}(e^{t(Z_1^2+Z_2^2)}) = \mathbb{E}(e^{tZ_1^2})\mathbb{E}(e^{tZ_2^2}) = \left(\frac{\frac{1}{4}}{\frac{1}{4}-t}\right)^{\frac{1}{2}}\left(\frac{\frac{1}{4}}{\frac{1}{4}-t}\right)^{\frac{1}{2}} = \left(\frac{\frac{1}{4}}{\frac{1}{4}-t}\right) \\
&\Rightarrow Z^2 \sim Gamma\Big(1,\frac{1}{4}\Big)
\end{aligned}$$