Università degli studi Roma Tre - Corso di Laurea in Matematica
Tutorato di ST1 - A.A. 2006/2007
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Soluzioni del tutorato n.1 del 6/3/2007

Esercizio 1.

(a) Sappiamo che è $Cov(X,Y) = \mathbb{E}((X-\mathbb{E}(X))(Y-\mathbb{E}(Y))) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. Calcoliamo quindi le distribuzioni marginali di X e Y. Riscriviamo la densità congiunta:

$$f_{X,Y}(x,y) = 2 \cdot \mathbb{1}_{(0,y)}(x)\mathbb{1}_{(0,1)}(y) = 2 \cdot \mathbb{1}_{(0,1)}(x)\mathbb{1}_{(x,1)}(y).$$

Si ha:

$$f_X(x) = \int_{-\infty}^{+\infty} 2 \cdot \mathbb{1}_{(0,1)}(x)\mathbb{1}_{(x,1)}(y)\,\mathrm{d}y = \int_x^1 2 \cdot \mathbb{1}_{(0,1)}(x)\,\mathrm{d}y = (2-2x)\mathbb{1}_{(0,1)}(x)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} 2 \cdot \mathbb{1}_{(0,y)}(x)\mathbb{1}_{(0,1)}(y)\,\mathrm{d}x = \int_0^y 2 \cdot \mathbb{1}_{(0,1)}(y)\,\mathrm{d}x = 2y\mathbb{1}_{(0,1)}(y)$$

$$\mathbb{E}(X) = \int_0^1 x(2-2x)\,\mathrm{d}x = \frac{1}{3}$$

$$\mathbb{E}(Y) = \int_0^1 2y^2\,\mathrm{d}y = \frac{2}{3}$$

$$Cov(X,Y) = \int_0^1 \mathrm{d}y \int_0^y \mathrm{d}x\, 2\big(x-\frac{1}{3}\big)\big(y-\frac{2}{3}\big) = \frac{1}{36}$$

(b)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{2 \cdot \mathbb{1}_{(0,1)}(x)\mathbb{1}_{(x,1)}(y)}{(2-2x)\mathbb{1}_{(0,1)}(x)} = \frac{1}{1-x}\mathbb{1}_{(x,1)}(y)$$

Esercizio 2. Sappiamo che:

$$f_Y(y) = \frac{e^{-\lambda}\lambda^y}{y!}\mathbb{1}_{\{0,1,\dots\}}(y)$$

$$f_{X|Y}(x|y) = \begin{cases} \binom{y}{x} p^x(1-p)^{y-x}\mathbb{1}_{\{0,1,\dots,y\}}(x) & \text{se } y>0 \\ \mathbb{1}_{\{0\}}(x) & \text{se } y=0 \end{cases}$$

$$f_{X,Y}(x,y) = e^{-\lambda}\mathbb{1}_{\{0\}}(x)\mathbb{1}_{\{0\}}(y) + \frac{e^{-\lambda}\lambda^y}{y!}\binom{y}{x}p^x(1-p)^{y-x}\mathbb{1}_{\{1,2,\dots\}}(y)\mathbb{1}_{\{0,1,\dots,y\}}(x) =$$

$$= e^{-\lambda}\mathbb{1}_{\{0\}}(x)\mathbb{1}_{\{0\}}(y) + \frac{e^{-\lambda}\lambda^y}{y!}\binom{y}{x}p^x(1-p)^{y-x}\mathbb{1}_{\{0,1,\dots\}}(x)\mathbb{1}_{\{x,\dots\}}(y)\mathbb{1}_{\{y\neq 0\}}$$

$$
\begin{aligned}
f_X(x) &= \sum_{y=0}^{+\infty} f_{X,Y}(x,y) = e^{-\lambda}\mathbb{1}_{\{0\}}(x) + \sum_{y=1}^{+\infty} \frac{e^{-\lambda}\lambda^y}{y!}\binom{y}{x} p^x(1-p)^{y-x}\mathbb{1}_{\{0,\dots\}}(x)\mathbb{1}_{\{x,\dots\}}(y) = \\
&= e^{-\lambda}\mathbb{1}_{\{0\}}(x) + \sum_{y=1}^{+\infty} \frac{e^{-\lambda}\lambda^y}{y!}\frac{y!}{x!(y-x)!} p^x(1-p)^{y-x}\mathbb{1}_{\{0\}}(x)\mathbb{1}_{\{1,\dots\}}(y) + \\
&\quad + \sum_{y=1}^{+\infty} \frac{e^{-\lambda}\lambda^y}{y!}\frac{y!}{x!(y-x)!} p^x(1-p)^{y-x}\mathbb{1}_{\{1,\dots\}}(x)\mathbb{1}_{\{x,\dots\}}(y) = \\
&= \sum_{y=0}^{+\infty} \frac{e^{-\lambda}\lambda^y}{y!}(1-p)^y\mathbb{1}_{\{0\}}(x) + \sum_{y=x}^{+\infty} \frac{e^{-\lambda}\lambda^y}{x!(y-x)!} p^x(1-p)^{y-x}\mathbb{1}_{\{1,\dots\}}(x) = \\
&= \sum_{y=0}^{+\infty} \frac{e^{-\lambda}(\lambda(1-p))^y}{y!}\frac{e^{-\lambda(1-p)}}{e^{-\lambda(1-p)}}\mathbb{1}_{\{0\}}(x) + \frac{e^{-\lambda}p^x}{x!}\mathbb{1}_{\{1,\dots\}}(x)\sum_{k=0}^{+\infty}\frac{\lambda^{k+x}(1-p)^k}{k!} = \\
&= e^{-\lambda p}\mathbb{1}_{\{0\}}(x) + \frac{e^{-\lambda}(\lambda p)^x}{x!}\mathbb{1}_{\{1,\dots\}}(x)\sum_{k=0}^{+\infty}\frac{(\lambda(1-p))^k}{k!}\frac{e^{-\lambda(1-p)}}{e^{-\lambda(1-p)}} = \\
&= e^{-\lambda p}\mathbb{1}_{\{0\}}(x) + \frac{e^{-\lambda p}(\lambda p)^x}{x!}\mathbb{1}_{\{1,\dots\}}(x) = \frac{e^{-\lambda p}(\lambda p)^x}{x!}\mathbb{1}_{\{0,1,\dots\}}(x) \sim Po(\lambda p)
\end{aligned}
$$

Esercizio 3. Siano

$$
X_1 = \begin{cases} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{cases} \qquad X_2 = \begin{cases} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{cases} \qquad X_3 = \begin{cases} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{cases}
$$

le tre monete (1 indica la realizzazione di una "testa" e 0 indica la realizzazione di una "croce") e definiamo

$$
X = X_1 + X_2 \qquad Y = 2 - X_2 - X_3
$$

Calcoliamo le marginali:

$$
\begin{aligned}
\mathbb{P}(X=0) &= \mathbb{P}(X_1+X_2=0) = \sum_{i=0}^{2}\mathbb{P}(X_1+X_2=0|X_2=i)\mathbb{P}(X_2=i) = \\
&= \mathbb{P}(X_1=0)\mathbb{P}(X_2=0) = \frac{1}{2}\frac{1}{2} = \frac{1}{4} \\
\mathbb{P}(X=1) &= \mathbb{P}(X_1+X_2=1) = \sum_{i=0}^{2}\mathbb{P}(X_1+X_2=1|X_2=i)\mathbb{P}(X_2=i) = \\
&= \mathbb{P}(X_1=1)\mathbb{P}(X_2=0) + \mathbb{P}(X_1=0)\mathbb{P}(X_2=1) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2} = \frac{1}{2} \\
\mathbb{P}(X=2) &= \mathbb{P}(X_1=1)\mathbb{P}(X_2=1) = \frac{1}{4} \\
\mathbb{P}(Y=0) &= \mathbb{P}(2-X_2-X_3=0) = \mathbb{P}(X_2=1)\mathbb{P}(X_3=1) = \frac{1}{2}\frac{1}{2} = \frac{1}{4}
\end{aligned}
$$

$$
\mathbb{P}(Y=1) = \frac{1}{2} \qquad\qquad \mathbb{P}(Y=2) = \frac{1}{4}
$$

$$\mathbb{P}(X=0,Y=0)=\sum_{i=0}^{2}\mathbb{P}(X_1+X_2=0,2-X_2-X_3=0|X_2=i)\mathbb{P}(X_2=i)=$$
$$=\mathbb{P}(X_1=0,2-X_3=0)\mathbb{P}(X_2=0)+$$
$$+\mathbb{P}(X_1+1=0,1-X_3=0)\mathbb{P}(X_2=1)+$$
$$+\mathbb{P}(X_1+2=0,-X_3=0)\mathbb{P}(X_2=2)=0$$
$$\mathbb{P}(X=0,Y=1)=\mathbb{P}(X_1=0,X_3=1)\mathbb{P}(X_2=0)=\frac{1}{2}\frac{1}{2}\frac{1}{2}=\frac{1}{8}$$
$$\mathbb{P}(X=0,Y=2)=\mathbb{P}(X_1=0,X_3=1)\mathbb{P}(X_2=0)=\frac{1}{2}\frac{1}{2}\frac{1}{2}=\frac{1}{8}$$

$$\mathbb{P}(X=1,Y=0)=\frac{1}{8}\qquad\mathbb{P}(X=1,Y=1)=\frac{1}{4}\qquad\mathbb{P}(X=1,Y=2)=\frac{1}{8}$$
$$\mathbb{P}(X=2,Y=0)=\frac{1}{8}\qquad\mathbb{P}(X=2,Y=1)=\frac{1}{8}\qquad\mathbb{P}(X=2,Y=2)=0$$

$$\mathbb{P}(Y=0|X=1)=\frac{\mathbb{P}(Y=0,X=1)}{\mathbb{P}(X=1)}=\frac{\frac{1}{8}}{\frac{1}{2}}=\frac{1}{4}$$
$$\mathbb{P}(Y=1|X=1)=\frac{\mathbb{P}(Y=1,X=1)}{\mathbb{P}(X=1)}=\frac{\frac{1}{4}}{\frac{1}{2}}=\frac{1}{2}$$
$$\mathbb{P}(Y=2|X=1)=\frac{\mathbb{P}(Y=2,X=1)}{\mathbb{P}(X=1)}=\frac{\frac{1}{8}}{\frac{1}{2}}=\frac{1}{4}$$

$$\mathbb{E}(X)=\frac{1}{2}+\frac{1}{2}=1\qquad\mathbb{E}(Y)=1$$
$$Cov(X,Y)=\mathbb{E}(XY)-1=\frac{1}{4}+2\frac{1}{8}+2\frac{1}{8}-1=-\frac{1}{4}$$